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Instanton Counting and qq-Characters

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Abstract

This thesis is a detailed account of instanton counting, a technique that applies equivariant localization to transform the instanton partition function of an $\mathcal{N} = 2$ supersymmetric gauge theory. The partition function is initially given in terms of equivariant integrals over resolved instanton moduli spaces and is transformed to a sum over partitions. We include a chapter on equivariant cohomology to provide the necessary mathematical background and make the treatment self-contained. In the last chapter, we discuss more recent developments in this area of theoretical physics, namely non-perturbative Dyson–Schwinger equations and qq-characters, with particular emphasis on their connections to representation theory.

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Contents

1 Introduction

Instanton counting, developed by Nikita Nekrasov in [Nek03], is a technique of evaluating the instanton partition function $\mathcal{Z}_{\text{inst}}$ of an $\mathcal{N} = 2$ supersymmetric $U(n)$ or $SU(n)$ gauge theory. Nekrasov used the formula for $\mathcal{Z}_{\text{inst}}$ obtained by instanton counting to give an expression of the Seiberg–Witten prepotential [SW94; Nek03], which is the limit $\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \log \mathcal{Z}_{\text{inst}}(\mathfrak{a}, \varepsilon_1, \varepsilon_2)$. Our objective with this thesis is to give a detailed account of instanton counting and to present and clarify the more recent developments [Nek16] of non-perturbative Dyson–Schwinger equations and qq-characters in the setting of quiver gauge theories.

We start in Chapter 2 by introducing the mathematical background on equivariant cohomology, characteristic classes and K-theory necessary for a treatment of instanton counting.

In Chapter 3 we will initially discuss some gauge theory background and define and compare various notions of instanton moduli spaces. We then perform instanton counting, that is, equivariant localization on the resolved moduli spaces, for pure $U(n)$ gauge theories and for quiver gauge theories – both in cohomology and in K-theory.

In Chapter 4 we will apply a non-perturbative transformation property of the measure obtained from instanton counting to construct specific quiver gauge theory observables known as qq-characters – a class of random rational functions. These qq-characters satisfy non-perturbative Dyson–Schwinger equations, meaning that they have non-singular expectations. They admit a geometric definition by which we try to relate them to representation theory, conjecturing a concrete relationship to the q-characters of quantum groups.

After the conclusion, we include two appendices: Appendix A on partitions and Young diagrams is best read before Section 3.2, where these concepts are first applied. Appendix B covers Dyson–Schwinger equations in quantum field theory and random matrix theory. While not essential for understanding the main text, it offers context and explains the terminology behind the non-perturbative Dyson–Schwinger equations.

Instanton counting

Our starting point will be the definition of the instanton partition function,

$$\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathfrak{M}(n,k)} 1, \tag{1}$$

as a generating function of equivariant integrals over the resolved moduli spaces of instantons, $\mathfrak{M}(n,k)$, first defined by Nakajima [Nak94b]. The integral $\int_{\mathfrak{M}(n,k)} 1$ is the equivariant integral of the identity element in the equivariant cohomology ring $H_T^\bullet(\mathfrak{M}(n,k))$. Here T is a torus group acting on the moduli space $\mathfrak{M}(n,k)$. Unlike ordinary cohomology, equivariant cohomology accounts for this group action. It will be the topic of Chapter 2.

In Chapter 3, we will show how to compute the integral $\int_{\mathfrak{M}(n,k)} 1$ by applying the equivariant localization theorem 2.15. This transforms the integral into a sum over the set of T-fixed points in $\mathfrak{M}(n,k)$. We will identify the fixed points with n-colored partitions $\boldsymbol{\lambda} \in \mathfrak{P}(n,k)$ of size k where by n-colored partition of size k we mean a vector $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^n)$ of n partitions whose sizes sum to $\sum_{\alpha=1}^{n} |\lambda^\alpha| = k$. Instanton counting is the process of applying equivariant localization to the integrals in (1) to transform the partition function into the infinite sum

$$\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \sum_{\boldsymbol{\lambda} \in \mathfrak{P}(n,k)} Z_{\boldsymbol{\lambda}}, \tag{2}$$

where $Z_{\boldsymbol{\lambda}}$ is determined by the T-representation structure of the tangent space to $\mathfrak{M}(n, k)$ at the fixed point parametrized by $\boldsymbol{\lambda}$. In the form (2), we also refer to $\mathcal{Z}_{\text{inst}}$ as the Nekrasov partition function. Because we integrate in equivariant cohomology, $\mathcal{Z}_{\text{inst}}$ (and also the summands $Z_{\boldsymbol{\lambda}}$) is a function of the equivariant parameters, by which we mean the coordinates on the Lie algebra $\mathfrak{t}$ of the torus T. These are denoted by $\mathfrak{a} \in \mathbb{C}^n$, $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ and physically interpreted as the Coulomb moduli and the parameters of the Ω-background. The partition function is then a function $\mathcal{Z}_{\text{inst}}(\mathfrak{a}, \varepsilon_1, \varepsilon_2)$.

We will perform instanton counting for a more general class of gauge theories, the quiver gauge theories. These are determined by three pieces of information: First there is the quiver, that is, a directed graph γ. The gauge group is determined by a coloring $\underline{n}$ assigning an integer n_i to each of the quiver's vertices. This defines the gauge group $G = \bigtimes_i U(n_i)$ of the quiver gauge theory. Unlike the massless theory in 1, quiver gauge theories contain matter fields. The matter fields are described by a second coloring $\underline{m}$. The instanton partition function of a quiver gauge theory will again be defined as the generating function of certain equivariant integrals over the theory's resolved instanton moduli spaces. Here we integrate not the equivariant cohomology ring's identity, but rather the equivariant Euler class of a vector bundle $\mathbf{B}$ over the moduli space, defined in terms of the quiver and the matter content $\underline{m}$.

Non-perturbative Dyson–Schwinger equations and qq-characters

In quantum field theory and random matrix theory, Dyson–Schwinger equations are essentially integration by parts formulas applied to the partition function. These integration by parts formulas can be seen as a consequence of translational symmetry of the integration measure (more generally; parametrization invariance of the integral), as we discuss in the appendix B.

The measure defined by the Nekrasov partition function

$$\mathcal{Z}_{\text{inst}} = \sum_{\lambda \text{ a partition}} \mu_\lambda, \qquad \mu_\lambda = \mathfrak{q}^{|\lambda|} Z_\lambda,$$

(here in the simplest case, with only one random partition) has a similar property. As the set that the measure is defined on is discrete, there is no notion of continuous translations on it. Rather, we consider non-perturbative transformations, where a partition λ's size gets increased by one, by adding a box to its Young diagram, thus turning it into another partition λ_+. The non-perturbative Dyson–Schwinger equations arise from the transformation property of the measure under this non-perturbative transformation:

$$\frac{\mu_{\lambda_+}}{\mu_\lambda} = -\mathfrak{q} \lim_{x \to x_+} \frac{1}{\mathcal{Y}(x)[\lambda]\mathcal{Y}(x+\varepsilon)[\lambda_+]} \tag{3}$$

for certain rational functions $\mathcal{Y}(x)$ which are random by their dependence on λ, and a particular $x_+ \in \mathbb{C}$ given by the box we added to the partition λ. One of the $\mathcal{Y}$-functions in the denominator of 3 has a zero at x_+, and the other has a pole there. The non-perturbative transformation of the measure 3 then tells us that these cancel each other. From this we can show that certain Laurent polynomials of the functions $\mathcal{Y}(x)$

are non-singular. The simplest example is obtained from (3) by rearranging to

$$\operatorname*{Res}_{x=x_+} \mu_{\lambda_+} \mathcal{Y}(x+\varepsilon)[\lambda_+] = -\operatorname*{Res}_{x=x_+} \frac{\mu_\lambda \mathfrak{q}}{\mathcal{Y}(x_+)[\lambda]}.$$

Using the fact that the poles of $\mathcal{Y}$-functions correspond to boxes in Young diagrams, they may be desingularized by summing over λ and λ_+, to yield an equation for expectations (cf. Proposition 4.3):

$$\left\langle \mathcal{Y}(x+\varepsilon) + \frac{\mathfrak{q}}{\mathcal{Y}(x)} \right\rangle = \text{polynomial in } x$$

Equations of this type, arising from transformation properties like (3) (which are more complicated for other quiver gauge theories) and first introduced by Nekrasov [Nek16] are called non-perturbative Dyson–Schwinger equations and will be a main focus of Chapter 4. The random Laurent polynomials in $\mathcal{Y}$-functions (like $\mathcal{Y}(x+\varepsilon) + \frac{\mathfrak{q}}{\mathcal{Y}(x)}$) satisfying Dyson–Schwinger equations, i.e. having non-singular expectation, are called qq-characters. They admit a geometric definition in terms of integrals over quiver varieties as well as potentially an algorithmic definition based on adding $\mathcal{Y}$-functions to cancel the poles of other $\mathcal{Y}$-functions.

The qq-characters present a remarkable connection between quiver gauge theory and representation theory: When a theory's quiver is a simply-laced Dynkin diagram (i.e. of type ADE), we associate to it a simple Lie algebra. This Lie algebra can be deformed (in two ways, cf. [Dri85; Jim85]) to a quantum group – which is not actually a group but rather a Hopf algebra. To study the representation theory of the quantum group, Knight [Kni95] and Frenkel–Reshetikhin [FR99] introduced the q-characters, generalizing the ordinary notion of a representation's character. The qq-characters, which arise in quiver gauge theory a priori without any direct links to representation theory, are believed to be a deformation of the q-characters of suitable representations. We will give a precise formulation to this conjecture and lay the foundations of a possible proof using the geometric definition of q- and qq-characters in terms of the equivariant topology of quiver varieties.

2 Equivariant cohomology, characteristic classes and localization

In the category of G-spaces, i.e. topological spaces with an action by a topological group G, G-equivariant cohomology is a natural cohomology theory accounting for this group action. It originates in the work of Armand Borel [Bor53] and Henri Cartan in the 1950s. In the 1980s, equivariant localization was developed by Duistermaat–Heckman [DH82], Berline–Vergne [BV82] and Atiyah–Bott [AB84]. It can be used to simplify integrals over manifolds with torus actions to sums over their fixed points.

Equivariant cohomology's significance in physics derives from using equivariant localization to simplify path integrals. The first such application was by Witten in [Wit82]. A comprehensive overview of the use of localization in quantum field theories is [Pes+17].

In this section we present the two main constructions of equivariant cohomology theories, the Borel construction which is for general topological groups and spaces, and the Cartan model which is an analogue to de Rham cohomology for manifolds with Lie group actions. We will also introduce equivariant characteristic classes of equivariant vector bundles which we then use in our discussion of equivariant localization. One of this thesis's aims is to retrace Nekrasov's procedure of instanton counting [Nek03] which is the application of equivariant localization to the path integral of an $\mathcal{N} = 2$ supersymmetric gauge theory. That will be the goal of Chapter 3.

2.1 The Borel construction

Throughout this subsection, X is a topological space and G a topological group acting continuously on X. Equivariant cohomology is meant to capture the topology of the space modulo the the group action. A naive approach to this would be to simply study the orbit space X/G. This, however, usually doesn't have the properties one would like it to have (for example the orbit space may not be Hausdorff, locally compact, compact, or a manifold, even when X is), so it is replaced by a ''homotopically correct'' version, called the homotopy quotient or Borel construction.

Recall that a topological space is called weakly contractible if all its homotopy groups are trivial. By Whitehead's theorem [Hat01, Theorem 4.5] weakly contractible CW complexes are contractible.

Definition 2.1. *For a topological group G, a* universal bundle $EG \to BG$ *is a principal G-bundle with weakly contractible total space EG. In this case BG is called a classifying space for G.*

It is named so because for finite CW complexes X the pullback f^*EG by maps $f : X \to BG$ defines a bijection between isomorphism classes of principal G-bundles over X and homotopy classes of maps $X \to BG$ [Ste99, §19.4]. In [Mil56] Milnor gave a construction of universal bundles for arbitrary topological groups.

As EG is a principal bundle, the fiber-preserving right action $G \curvearrowright EG$ (part of the structure of the principal bundle) on the total space is free. Hence, defining a left action on the product space $EG \times X$ by

$$g \cdot (e, x) \coloneqq (eg^{-1}, gx),$$

(here e is an element of EG, not the neutral element of G) this action is free as well.

Definition 2.2. *The homotopy quotient of X by G is the orbit space*

$$X_G \coloneqq (EG \times X)/G$$

with the G-action on $EG \times X$ defined as above.

This definition depends on the choice of EG. However, two different universal bundles result in weakly homotopy equivalent homotopy quotients [TA20, Theorem 4.10].

If EG is a CW complex, then by Whitehead's theorem it is contractible, hence $EG \times X$ is homotopy equivalent to X. In this way we interpret X_G as a homotopically correct version of the quotient X/G and as such it serves to define the equivariant cohomology of G-space X.

Definition 2.3. *For a G-space X the G-equivariant cohomology with coefficients in a ring R is defined as the (singular) cohomology of the homotopy quotient X_G:*

$$H^\bullet_G(X; R) \coloneqq H^\bullet(X_G; R)$$

Technically this definition depends on the choice of universal bundle EG. However, the weak homotopy equivalence between the homotopy quotients with respect to two different universal bundles induces an isomorphism on cohomology. As with ordinary cohomology theories, the G-equivariant cohomology of X is a module over its coefficient ring $H^\bullet_G(*)$, the equivariant cohomology of the one-point space. The module structure comes from the unique map $X \to *$ inducing a ring homomorphism $H^\bullet_G(*) \to H^\bullet_G(X)$. The one-point space's homotopy quotient $*_G = * \times EG/G = BG$ is exactly the classifying space of the group G.

In the special case where the action $G \curvearrowright X$ is free, the homotopy quotient X_G is a fiber bundle over the quotient X/G with fiber EG. As EG is contractible, this is homotopy equivalent to X/G. So in this case, the equivariant cohomology of X is just the cohomology of the quotient X/G.

2.2 The Cartan model

Among the many different cohomology theories on smooth manifolds, de Rham cohomology, defined in terms of the complex of differential forms connected by the exterior derivative, plays a central role. In the case of equivariant cohomology, a similar construction in terms of differential forms exists, called the Cartan model. Throughout, X will be a smooth manifold with a smooth action by a Lie group G with real Lie algebra $\mathfrak{g}$. The case of complex Lie groups with complex Lie algebras will be discussed at the end of the subsection.

The differential complex defining the Cartan model is constituted of so-called G-equivariant differential forms on M. To define G-equivariant forms and the differential connecting the cochain groups, we must first introduce the graded algebra of polynomials on the Lie algebra $\mathfrak{g}$. This is easily identified with the symmetric (tensor) algebra $S^\bullet(\mathfrak{g}^\vee)$ of the dual space $\mathfrak{g}^\vee$ to $\mathfrak{g}$. Now, G-equivariant differential forms on the G-manifold M will be certain polynomials on $\mathfrak{g}$ taking their values not in the scalar field but instead in the graded algebra of differential forms on M. The space of these $\Omega^\bullet(M)$-valued polynomials is

$$S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M).$$

The Lie group G acts on the Lie algebra $\mathfrak{g}$ by the adjoint action Ad. The dual to this action, on $\mathfrak{g}^\vee$, naturally extends to an action on the polynomial algebra $S^\bullet(\mathfrak{g}^\vee)$. Furthermore, G acts on the differential forms $\Omega^\bullet(M)$ through the pushforward; regarding $g \in G$ as a diffeomorphism of M we may define $g \cdot \alpha = g_*\alpha$. Having defined natural actions of G on $S^\bullet(\mathfrak{g}^\vee)$ and $\Omega^\bullet(M)$, we also have an action on the tensor product, $G \curvearrowright S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$ and may now define G-equivariant differential forms:

Definition 2.4. *A G-equivariant differential form α on M is an element of $S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$ that is invariant under the action of G. We denote the algebra of G-equivariant differential forms by*

$$\Omega_G^\bullet(M) \coloneqq (S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M))^G.$$

G-equivariant forms are not to be confused with G-invariant forms which are ordinary differential forms on M (not form-valued polynomials) that are invariant under the group action, that is, elements of $\Omega^\bullet(M)^G$. An equivariant differential form is exactly a polynomial $\alpha \in S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$ for which

$$\alpha(\mathrm{Ad}_g X) = g_* \alpha(X) \tag{4}$$

for all $g \in G$ and $X \in \mathfrak{g}$, i.e. the map $\alpha : \mathfrak{g} \longrightarrow \Omega^\bullet(M)$ is equivariant.

To define cohomology using this algebra of equivariant differential forms, we should first discuss its grading. On the algebra of differential forms $\Omega^\bullet(M)$ we use the usual grading, whereas on $S^\bullet(\mathfrak{g}^\vee)$ we rescale the grading by a factor of two, meaning that for monomials $p = \eta_1 \dots \eta_k$ in $S^\bullet(\mathfrak{g}^\vee)$ we define $\deg(p) = 2k$. These two gradings are invariant under the action of G and thus define a bigrading on $\Omega_G^\bullet(M)$. The total grading on $\Omega_G^\bullet(M)$ is then given by

$$\Omega_G^k(M) = \bigoplus_{2i+j=k} (S^i(\mathfrak{g}^\vee) \otimes \Omega^j(M))^G.$$

Elements of $\Omega_G^k(M)$ are called G-equivariant k-forms.

The integration of equivariant differential forms is defined in the obvious way:

Definition 2.5. *The integral $\int_M \alpha \in S^\bullet(\mathfrak{g}^\vee)$ of an equivariant form $\alpha \in \Omega_G^\bullet(M)$ over a compact oriented G-manifold M is defined as the polynomial*

$$\left(\int_M \alpha\right)(X) \coloneqq \int_M \alpha(X)$$

for $X \in \mathfrak{g}$.

Unlike for ordinary differential forms, the integral $\int_M \alpha$ is not a scalar but rather a polynomial on the Lie algebra $\mathfrak{g}$. Note that

$$\left(\int_M \alpha\right)(\mathrm{Ad}_g X) = \int_M \alpha(\mathrm{Ad}_g X) = \int_M g_* \alpha(X) = \int_M \alpha(X)$$

when the Lie group is connected. Thus, in that case, the integral is a G-invariant polynomial on $\mathfrak{g}$. We will show in remark 2.8 that, in the case where G is additionally compact, the space of G-invariant polynomials on $\mathfrak{g}$ is isomorphic to the equivariant cohomology of the one-point space. We may then regard $\int_M \alpha$ as an element of $H_G^\bullet(*)$.

Next we introduce the differential connecting the cochain groups, the analogue of the differential d of ordinary differential forms. For this we need the notion of interior multiplication of differential forms $\alpha \in \Omega^\bullet(M)$ by Lie algebra elements X: The infinitesimal version of the action $G \curvearrowright M$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ to the Lie algebra of vector fields. Thus, elements $X \in \mathfrak{g}$ are mapped to vector fields $\overline{X}$ on M, for which we can take the interior product $\iota_{\overline{X}}\alpha$. The interior product lowers the degree of a non-equivariant differential form $\alpha \in \Omega^\bullet(M)$ by one. However, as a polynomial of $\mathfrak{g}$, $\iota_{\overline{X}}\alpha$

has degree one while α has degree zero. The polynomial degree is weighted double in the grading on $S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$, thus, overall, interior multiplication raises the degree of form-valued polynomials by one.

Definition 2.6. *The equivariant exterior derivative is the linear map $D : \Omega^\bullet_G(M) \to \Omega^\bullet_G(M)$ defined on an $\Omega^\bullet(M)$-valued polynomial $\alpha \in \Omega^\bullet_G(M) = (S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M))^G$ as*

$$(D\alpha)(X) \coloneqq d(\alpha(X)) - \iota_{\overline{X}}(\alpha(X))$$

where X is an element of $\mathfrak{g}$ on which we evaluate the polynomial.

As both the exterior derivative and interior multiplication raise the degree of equivariant forms by one, so does the equivariant exterior derivative. Now we check that D makes $\Omega^\bullet_G(M)$ a cochain complex:

$$\begin{aligned}(D^2\alpha)(X) &= D(D\alpha)(X) = d((D\alpha)(X)) - \iota_{\overline{X}}((D\alpha)(X)) \\ &= d\big(d(\alpha(X)) - \iota_{\overline{X}}(\alpha(X))\big) - \iota_{\overline{X}}\big(d(\alpha(X)) - \iota_{\overline{X}}(\alpha(X))\big) \\ &= -d\iota_{\overline{X}}(\alpha(X)) - \iota_{\overline{X}}d(\alpha(X)) = -\mathcal{L}_{\overline{X}}(\alpha(X))\end{aligned}$$

by Cartan's homotopy formula. By (4), $\mathcal{L}_{\overline{X}}\alpha(Y) = \alpha(\mathrm{ad}_X Y)$, which vanishes for $Y = X$. The cochain complex

$$\Omega^0_G(M) \xrightarrow{D} \Omega^1_G(M) \xrightarrow{D} \dots \xrightarrow{D} \Omega^n_G(M)$$

then defines the cohomology $H^\bullet(\Omega^\bullet_G(M), D)$. We define the wedge product on equivariant differential forms by

$$(\alpha \otimes p) \wedge (\omega \otimes q) \coloneqq (\alpha \wedge \omega) \otimes (p \cdot q).$$

This is graded-commutative and descends to cohomology, making $H^\bullet(\Omega^\bullet_G(M), D)$ a ring.

There is a version of the de Rham theorem for equivariant cohomology and the Cartan model as well:

Theorem 2.7. *For any compact, connected Lie group G acting smoothly on a manifold M, the cohomology of the Cartan model is canonically isomorphic to the equivariant cohomology with real coefficients:*

$$H^\bullet(\Omega^\bullet_G(M), D) \cong H^\bullet_G(M; \mathbb{R})$$

The equivariant de Rham theorem holds for complex coefficients as well: Working with complex-valued differential forms amounts to complexifying the cochain complex $(\Omega^\bullet_G(M) \otimes \mathbb{C}, D \otimes \mathbb{1}_\mathbb{C})$. We then have $H^\bullet(\Omega^\bullet_G(M) \otimes \mathbb{C}, D \otimes \mathbb{1}_\mathbb{C}) = H^\bullet(\Omega^\bullet_G(M); D) \otimes \mathbb{C} \cong H^\bullet_G(M; \mathbb{R}) \otimes \mathbb{C} \cong H^\bullet_G(M; \mathbb{C})$.

Remark 2.8. When G is compact connected, we can apply the de Rham theorem to computing of the equivariant cohomology of the one-point space $*$. Clearly $\Omega^\bullet(*) = \Omega^0(*) = \mathbb{R}$. Thus, $\Omega^\bullet_G(M) = (S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(*))^G = S^\bullet(\mathfrak{g}^\vee)^G$, meaning the Cartan complex is exactly the G-invariant polynomials on the Lie algebra $\mathfrak{g}$. The degrees of elements $p \in S^\bullet(\mathfrak{g}^\vee)^G$ are defined to be twice their polynomial degree, thus the complex $S^\bullet(\mathfrak{g}^\vee)^G$ is alternatingly zero (in odd degrees) and non-zero (in even degrees). Hence, its cohomology is exactly the complex itself:

$$H^\bullet_G(*) = S^\bullet(\mathfrak{g}^\vee)^G$$

In the case where the Lie group is a torus $T = U(1)^n$, this simplifies further to $H^\bullet_T(*) = S^\bullet(\mathfrak{t}^\vee)$ where $\mathfrak{t}$ is the torus Lie algebra, as T being abelian means that its adjoint action on $S^\bullet(\mathfrak{t}^\vee)$ is trivial.

Complex Lie groups. In instanton counting we perform equivariant localization with respect to an action by a complex (non-compact) torus. For this reason we should also discuss the Cartan model for complex Lie groups. For a complex Lie group $G_\mathbb{C}$, the Lie algebra $\mathfrak{g}_\mathbb{C}$ is a complex vector space. Complex-valued equivariant differential forms may then be realized as complex polynomials $\mathfrak{g}_\mathbb{C} \to \Omega^\bullet(M;\mathbb{C})$, rather than as real polynomials: The difference is the same as that between polynomials $\mathbb{C} \to \mathbb{C}$ and $\mathbb{R}^2 \to \mathbb{C}$. Then, we define

$$\Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}) \coloneqq (\Omega^\bullet(M;\mathbb{C}) \otimes S^\bullet(\mathfrak{g}_\mathbb{C}^\vee))^{G_\mathbb{C}}$$

where by $S^\bullet(\mathfrak{g}_\mathbb{C}^\vee)$ we mean the algebra of complex polynomials. The equivariant exterior derivative $D_c : \Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}) \to \Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C})$ is defined as in the case of the real Lie group. Integration $\int_M$: $\Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}) \to S^\bullet(\mathfrak{g}_\mathbb{C}^\vee)$ is defined in the obvious way:

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X).$$

Now suppose that $G_\mathbb{C}$ is the complexification of a compact connected real Lie group G (for example a compact torus). We may restrict $G_\mathbb{C}$-equivariant forms $\alpha : \mathfrak{g}_\mathbb{C} \to \Omega^\bullet(M;\mathbb{C})$ to the real subalgebra $\mathfrak{g}$. The result is a complex-valued G-equivariant form. Furthermore, any polynomial of $\mathfrak{g}_\mathbb{C}$ is fully determined by its restriction to $\mathfrak{g}$ so that this correspondence can be inverted. From this it is clear that

$$\Omega^\bullet(M;\mathbb{C}) \otimes S^\bullet(\mathfrak{g}_\mathbb{C}^\vee) \cong \Omega^\bullet(M;\mathbb{C}) \otimes S^\bullet(\mathfrak{g}^\vee)$$

as complex vector spaces. Moreover, an element of the LHS is $G_\mathbb{C}$-fixed if and only if the corresponding element on the RHS is G-fixed. Thus,

$$\Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}) \cong \Omega^\bullet_G(M;\mathbb{C}). \tag{5}$$

This isomorphism commutes with the equivariant exterior derivatives $D_c, D \otimes \mathbb{1}_\mathbb{C}$, so it induces an isomorphism on the Cartan model cohomologies; $H^\bullet(\Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}), D_c) \cong H^\bullet(\Omega_G(M;\mathbb{C}), D \otimes \mathbb{1}_\mathbb{C})$.

In [AF23, Sec. 3.2] it is shown that equivariant cohomology $H^\bullet_G(M)$ is functorial not just in the space M, but also in the group G, and that group homomorphisms that are homotopy equivalences induce isomorphisms on equivariant cohomology. In particular, as $G_\mathbb{C}$ deformation retracts to G,

$$H^\bullet_{G_\mathbb{C}}(M;\mathbb{C}) \cong H^\bullet_G(M;\mathbb{C}), \tag{6}$$

the isomorphism being induced by the inclusion $G \hookrightarrow G_\mathbb{C}$.

Ultimately, the equivariant de Rham theorem for G, together with (5) and (6) implies an equivariant de Rham theorem for $G_\mathbb{C}$:

Corollary 2.9. *Let G be a compact connected Lie group and $G_\mathbb{C}$ its complexification. If $G_\mathbb{C}$ acts smoothly on a compact manifold M, then there is a natural isomorphism*

$$H^\bullet(\Omega^\bullet_{G_\mathbb{C},c}(M;\mathbb{C}), D_c) \cong H^\bullet_{G_\mathbb{C}}(M;\mathbb{C})$$

Remark 2.8 has a version for complexifications of compact connected groups as well:

$$H^\bullet_{G_\mathbb{C}}(*;\mathbb{C}) \cong S^\bullet(\mathfrak{g}_\mathbb{C}^\vee)^{G_\mathbb{C}}$$

For non-compact tori $T_\mathbb{C} \cong (\mathbb{C}^\times)^n$, with Lie algebra $\mathfrak{t}_\mathbb{C} \cong \mathbb{C}^n$,

$$H^\bullet_{T_\mathbb{C}}(*;\mathbb{C}) \cong S^\bullet(\mathfrak{t}_\mathbb{C}^\vee) \cong S^\bullet(\mathfrak{t}^\vee) \otimes \mathbb{C}.$$

2.3 Equivariant characteristic classes

Characteristic classes like Stiefel–Whitney, Euler, Chern and Pontryagin classes can be generalized to the equivariant setting. We will need in particular the equivariant Euler classes, as they occur in the localization Theorem 2.14 in the next subsection. The theory of ordinary characteristic classes is discussed in [MS74]. Equivariant characteristic classes are defined for equivariant vector bundles, where there's a G-action on the total space compatible with that on the base space:

Definition 2.10. *A G-equivariant vector bundle over a topological G-space X is a (real or complex) vector bundle $\pi : E \to X$ with an action $G \curvearrowright E$ by linear maps between the fibers, for which the bundle projection π is an equivariant map, meaning that for all $g \in G$, $e \in E$,*

$$\pi(ge) = g\pi(e).$$

We now discuss characteristic classes for equivariant vector bundles. These will live in the equivariant cohomology $H^\bullet_G(X)$ so, as the equivariant cohomology is just the cohomology of the homotopy quotient X_G, it makes sense to define them as the ordinary characteristic classes of vector bundles over X_G associated to the original equivariant bundle $E \to X$. These bundles are again constructed by the Borel construction: It is not hard to see that for every equivariant vector bundle over a G-space X, the homotopy quotient $E_G \coloneqq (EG \times E)/G$ is a vector bundle over $X_G = (EG \times X)/G$ and these bundles have identical rank.

Definition 2.11. *For a complex equivariant vector bundle E over a G-space X, its equivariant Chern classes are*

$$c_k^G(E) \coloneqq c_k(E_G) \in H^{2k}(X_G;\mathbb{Z}) = H^{2k}_G(X;\mathbb{Z}).$$

The total equivariant Chern class is $c^G(E) = c_0^G(E) + c_1^G(E) + \cdots + c_n^G(E)$, where $n = \mathrm{rk}_\mathbb{C} E$.

While Chern classes are defined in integer coefficients cohomology, we generally care about their images in $H^\bullet_G(X;\mathbb{R})$ or $H^\bullet_G(X;\mathbb{C})$, so that, when X is a compact manifold, they may be identified with closed equivariant differential forms and integrated.

Equivariant Chern classes behave much like ordinary Chern classes: The zeroth class $c_0^G(E)$ is always 1. For direct sums $E \oplus F$, we have

$$c^G(E \oplus F) = c^G(E) \smile c^G(F)$$

and for equivariant bundle maps $f : E \to F$, they satisfy

$$f^* c^G(E) = c^G(f^* E).$$

Note, however, that while the higher non-equivariant Chern classes of a topologically trivial bundle are necessarily 0, this is not the case for equivariant bundles: Such a bundle would be the pullback π^*E of an equivariant bundle $E \to *$ by the collapsing map $\pi : X \to *$. In Section 2.5 we detail how to identify E with a representation of G and how the equivariant Chern classes are then determined by the weights of the representation. Thus, if the representation is non-trivial, the equivariant Chern classes $c_k(E)$ do not vanish and then neither do those of the topologically trivial bundle over X: $c_k^G(\pi^*E) = \pi^*c_k^G(E) \neq 0$.

Of course, for $k > n = \operatorname{rk} E$ the equivariant Chern class c_k^G must vanish as it does in the non-equivariant case. The top-degree equivariant Chern class is called the equivariant Euler class and we denote it by

$$\epsilon^G(E) \coloneqq c^G_{\operatorname{rk} E}(E).$$

The equivariant Euler class can also be defined for oriented real equivariant vector bundles, as the Euler class of the homotopy quotient bundle $E_G \to M_G$, which can be shown to inherit an orientation from E.

To introduce the equivariant Chern roots, we must first discuss the splitting principle (cf. [Hat03, Proposition 3.3]): For a finite-rank vector bundle $E \to X$ over a compact space, there exists a space $F(E)$ called the complete flag bundle and a map $p : F(E) \to X$ such that the pullback map $p^* : H^\bullet(X) \to H^\bullet(F(E))$ is injective and that

$$p^*E = L_1 \oplus \cdots \oplus L_n$$

splits as the direct sum of line bundles. In the case where E is complex, the Chern roots are then defined to be the first Chern classes of the line bundles; $c_1(L_1), \dots, c_1(L_n) \in H^2(F(E))$. The pullbacks of the Chern classes $p^*c_k(E)$ are then expressed as the elementary symmetric polynomials of the Chern roots:

$$p^*c_k(E) = \sigma_k(c_1(L_1), \dots, c_1(L_n)) \quad \text{where} \quad \sigma_k(X_1, \dots, X_n) \coloneqq \sum_{1 \le k_1 < \cdots < k_i \le n} X_{k_1} \dots X_{k_i}.$$

In particular, the Euler class is just the product of all Chern roots:

$$p^*e(E) = r_1(E) \dots r_n(E)$$

While the Chern roots in principle depend on the concrete choice of splitting $p^*E = L_1 \oplus \cdots \oplus L_n$, their symmetric polynomials are independent of this. Furthermore, by the fundamental theorem on symmetric polynomials [Mac98, (2.4)], any symmetric polynomial of $X_1, \dots, X_n$ can be uniquely expressed as a polynomial of the elementary symmetric polynomials σ_k. This theorem is the basis for defining other characteristic classes in terms of the Chern roots, even when the base space X is not compact (so that there technically are no Chern roots). For a finite-rank vector bundle $E \to X$ (over a possibly non-compact space), polynomials of the formal Chern roots $r_1(E), \dots, r_n(E)$ (which are merely variables, not elements of the cohomology) are thus interpreted as polynomials of the Chern classes $\sigma_k(E)$, making them elements of the cohomology ring $H^\bullet(X)$.

The Chern character $\operatorname{Ch}(E) \in \prod_{k=0}^\infty H^k(X; \mathbb{Z})$ of a complex vector bundle $E \to X$ is defined as

$$\operatorname{Ch}(E) \coloneqq \sum_{i=1}^n e^{r_i(E)},$$

which is a symmetric power series of the formal Chern roots. We explain how this characteristic class is expressed in terms of the Chern classes: Newton's identities express power sum polynomials $p_k(X_1, \dots,$

$X_n) = X_1^k + \cdots + X_n^k$ in terms of the elementary symmetric polynomials, e.g.

$$p_1 = \sigma_1, \quad p_2 = \sigma_1^2 - 2\sigma_2, \quad p_3 = \sigma_1^3 - 3\sigma_2\sigma_1 + 3\sigma_3, \quad \ldots$$

Writing $p_k = \nu_k(\sigma_1, \ldots, \sigma_k)$ for these identities, the components of the Chern character are given in terms of the Chern classes without reference to the formal Chern roots:

$$\mathrm{Ch}(E) = \sum_{k=0}^{\infty} \frac{1}{k!} \nu_k(c_1(E), \ldots, c_k(E)) \tag{7}$$

The Chern character is not just additive under direct sums but also multiplicative under tensor products:

$$\mathrm{Ch}(E \oplus F) = \mathrm{Ch}(E) + \mathrm{Ch}(F), \quad \mathrm{Ch}(E \otimes F) = \mathrm{Ch}(E) \smile \mathrm{Ch}(F)$$

Thus, in cases where the cohomology ring of X truncates, the Chern character defines a ring homomorphism from the K-theory of X to its cohomology [Hat03, Sec. 4.1].

For an equivariant complex vector bundle $E \to X$, the equivariant Chern character is again defined using the homotopy quotient:

$$\mathrm{Ch}^G(E) \coloneqq \mathrm{Ch}(E_G) \in \prod_{k=0}^{\infty} H_G^k(X; \mathbb{Z})$$

The relation (7) between Chern classes and character carry over to the equivariant case. In cases where the homotopy quotient X_G is compact, so that the splitting principle applies to E_G, one can define equivariant Chern roots as elements of $H^2(F(E_G); \mathbb{Z})$. In general, equivariant characteristic classes can be expressed as symmetric functions of the formal equivariant Chern roots, denoted by $r_i^G(E)$.

We will also make use of the equivariant Chern polynomial

$$c_x^G(E) \coloneqq \sum_{k=0}^{n} c_k^G(E) x^{n-k} = \prod_{i=1}^{n} \left(x + r_i^G(E)\right), \qquad x \in \mathbb{C}.$$

Our definition differs from another common convention, $\sum c_k^G x^k$. Our definition has the useful property that, in the limit $x = 0$, the polynomial is the equivariant Euler class; $c_x^G(E) = \epsilon^G(E)$. In other words, the equivariant Euler class is the polynomial's constant part. While the notation c_x^G for the equivariant Chern polynomial collides with the notation c_k^G for the equivariant Chern classes, this will pose no problem as we never use the Chern classes themselves.

Another characteristic class we will make need is the equivariant Todd class, defined in terms of the formal equivariant Chern roots as the symmetric power series

$$\mathrm{Td}^G(E) \coloneqq \prod_{i=1}^{n} \frac{r_i^G(E)}{1 - e^{-r_i^G(E)}}.$$

Lastly, we mention that there also exists an equivariant version of Chern–Weil theory, expressing equivariant characteristic classes as invariant polynomials of the curvature 2-form of an equivariant connection on the vector bundle. The equivalence of the two approaches was proved relatively late in the history of equivariant cohomology, in [BT01].

2.4 Torus representations

We gather some facts about representations of tori. For their proofs we refer to [Bum04, Chap. 15]. By the character of a finite-dimensional real or complex representation $\rho : G \to GL(V)$, we mean the map

$$\chi_V : G \to \mathbb{C}, \quad g \mapsto \mathrm{tr}(\rho(g)).$$

Definition 2.12. *A compact torus is a compact connected abelian Lie group T. A non-compact torus is the complexification $T_{\mathbb{C}}$ of a compact torus T.*

Complexifications are characterized up to isomorphism by a universal property and always exist for compact Lie groups (cf. [Bum04, Chap. 24]). A non-compact torus's Lie algebra is $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$.

For an r-dimensional compact torus T, the Lie algebra $\mathfrak{t}$ is an abelian group (like every vector space) and the exponential map $\exp : \mathfrak{t} \to T$ is a group homomorphism. It can be checked that $\ker(\exp)$ is a lattice in $\mathfrak{t}$. Then

$$T \cong \mathfrak{t}/\ker(\exp) \cong \mathbb{R}^r/\mathbb{Z}^r \cong U(1)^r.$$

For the non-compact torus $T_{\mathbb{C}}$ which is the complexification of T:

$$T_{\mathbb{C}} \cong (\mathbb{C}^\times)^r$$

Note that $\mathbb{C}^\times = GL(1, \mathbb{C})$.

Irreducible finite-dimensional complex representations of $T_{\mathbb{C}}$ are all one-dimensional and classified by their character which is a homomorphism (because the representations are 1-dimensional) $\chi_V : T_{\mathbb{C}} \to \mathbb{C}^\times$, i.e. a monomial $\chi_V(t_1, \dots, t_r) = t_1^{k_1} \dots t_r^{k_r}$ (using the isomorphism $T_{\mathbb{C}} \cong (\mathbb{C}^\times)^r$). This representation's weight $w : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ is $w(x_1, \dots, x_r) = \sum_i k_i x_i$. Irreducible complex representations of the complex torus T extend uniquely to irreducible representations of $T_{\mathbb{C}}$ and are thus also classified by their characters.

Irreducible real representations of T are also determined by their characters. There are two cases: First there's the one-dimensional trivial representation. The non-trivial irreducible representations are all two-dimensional; for each $\mathbf{k} \in \mathbb{Z}^r \setminus 0$ there is the representation $V_{\mathbf{k}}$ given by

$$e^{(i\theta_1, \dots, i\theta_r)} \mapsto \begin{pmatrix} \cos(\mathbf{k} \cdot \boldsymbol{\theta}) & -\sin(\mathbf{k} \cdot \boldsymbol{\theta}) \\ \sin(\mathbf{k} \cdot \boldsymbol{\theta}) & \cos(\mathbf{k} \cdot \boldsymbol{\theta}) \end{pmatrix} = \exp \begin{pmatrix} 0 & -\mathbf{k} \cdot \boldsymbol{\theta} \\ \mathbf{k} \cdot \boldsymbol{\theta} & 0 \end{pmatrix}, \tag{8}$$

where $\mathbf{k} \cdot \boldsymbol{\theta}$ denotes the dot product. Its eigenvalues are $e^{\pm i \mathbf{k} \cdot \boldsymbol{\theta}}$, so it complexifies to the representation with character $t_1^{k_1} \dots t_r^{k_r}$.

Suppose $E \to M$ is a (real or complex) G-equivariant vector bundle and $x \in M$ is a G-fixed point. Then $E_x \to \{x\}$ is a G-equivariant bundle over the one-point space $\{x\}$. This may equivalently be regarded as a representation of G, called the isotropy representation at x. We are particularly interested in the isotropy representations $T_x M$ on tangent spaces, induced by the equivariant tangent bundle $TM \to M$.

If V is a real T-representation, there exists a decomposition into irreducibles

$$V \cong \bigoplus_{\mathbf{k}} V_{\mathbf{k}}$$

where $V_{\mathbf{k}}$ is given in (8). If $\mathbf{k} \neq 0$, then $V_{\mathbf{k}}$ is two-dimensional and oriented by the T-action, after choosing an isomorphism $T \cong U(1)^r$. Thus, if all $\mathbf{k}$ are non-zero, then the representation V has an orientation depending only on the identification of T with $U(1)^r$.

Considering such a real T-representation with non-zero weights as a T-equivariant real vector bundle over $*$, we may consider its equivariant Euler class $\epsilon^T(V) \in H^\bullet_T(*)$. We refer to this as the representation's Euler class. Under the identification of $H^\bullet_T(*)$ with the polynomial algebra $S^\bullet(\mathfrak{t}^\vee)$, it has a simple, useful expression:

Lemma 2.13. *Let V be a complex T-representation. Then, under the isomorphism $H^\bullet_T(*;\mathbb{R}) \cong S^\bullet(\mathfrak{t}^\vee)$ from remark 2.8, its Euler class is mapped to the product of its weights, i.e.*

$$\epsilon^T(V) \mapsto \prod_{i=1}^n w_i \in S^\bullet(\mathfrak{t}^\vee),$$

(the product is over the multiset of weights, meaning the same weight can occur more than once) which one may view as the determinant of the Lie algebra representation.

This can be proved in equivariant Chern–Weil theory, using that the Euler class (top Chern class) is represented by $p(F_{A,T})$ where $F_{A,T}$ is the T-equivariant curvature of the bundle and $p(r_1, \ldots, r_n) = \prod r_i$ is a polynomial on $\mathfrak{t}$ (cf. [Pes+17, Sec. 2.5]). The lemma extends to $T_\mathbb{C}$ and the isomorphism $H^\bullet_{T_\mathbb{C}}(*;\mathbb{C}) \cong S^\bullet(\mathfrak{t}^\vee_\mathbb{C})$ by the comments above Corollary 2.9.

It is clear from the weight decomposition (this is the decomposition into irreducible subrepresentations) of a complex T- or $T_\mathbb{C}$-representation

$$V \cong \bigoplus_w V_w$$

(here the sum is over the multiset of weights, such that each weight space is one-dimensional) and Lemma 2.13 that the equivariant Chern roots of V are identified with the weights of the representation. Thus, the other equivariant Chern classes are also elementary symmetric polynomials of the weights, and the equivariant Chern character is

$$\mathrm{Ch}^T(V) = \sum_w e^w = \chi_V \circ \exp,$$

identifying it with the usual notion of character χ_V of the representation.

2.5 Equivariant localization

The instanton partition function is given as the integral of an equivariantly closed differential form over the resolved framed moduli space of instantons, which carries a torus action. In [Nek03], using localization, Nekrasov reduced this integral to a sum over infinitely many fixed points of the torus action in the moduli space. Much of the present thesis is devoted to rigorously presenting this process, instanton counting.

Theorem 2.14. *Let M be a compact oriented manifold with a smooth action of a compact torus T and denote by F the set of T-fixed points in M. Then, for any equivariantly closed form $\alpha \in \Omega^\bullet_T(M)$, we have*

$$\int_M \alpha = \int_F \frac{\iota_F^* \alpha}{\epsilon^T(NF)}$$

where $\iota_F : F \hookrightarrow M$ is the inclusion, NF the normal bundle of F and $\epsilon^T(NF)$ its equivariant Euler class.

To make sense of this, it should be noted that the set of T-fixed points F is a submanifold (cf. [TA20, Theorem 25.1]) and also equipped with a smooth T-action. The pullback $\iota_F^*\alpha$ is a G-equivariant form on

F. The normal bundle NF has to be oriented for the equivariant Euler class $\epsilon^T(NF)$ to be defined. The orientation is induced by the T-action; at a fixed point f, the tangent space $T_f M$ is a real T-representation. Its decomposition into irreducibles is such that the 1-dimensional irreducible representations are tangent to F and the 2-dimensional irreducible representations form the normal space. This induces an orientation on NF as discussed in Section 2.4.

As the T-action on F is trivial, the equivariant cohomology is $H^\bullet_T(F) = S^\bullet(\mathfrak{t}^\vee) \otimes H^\bullet(F)$. Then the Euler class has the form

$$\epsilon^T(NF) = p_m + p_{m-1}\alpha_1 + \cdots + \alpha_m$$

where $m = \operatorname{codim} F$, $p_i \in S^i(\mathfrak{t}^\vee)$, $\alpha_i \in H^i(F)$. In the localization of the $S^\bullet(\mathfrak{t}^\vee)$-module $H^\bullet_T(F)$ with respect to the polynomial p_m, the Euler class is then seen to be invertible:

$$\frac{1}{\epsilon^T(NF)} = \frac{1}{p_m}\left(1 + \frac{\alpha}{p_m} + \frac{\alpha^2}{p_m^2} + \cdots + \frac{\alpha^{q-1}}{p_m^{q-1}}\right)$$

where $\alpha = -(p_{m-1}\alpha_1 + \cdots + \alpha_m)$. The equivariant integral $\int_F : H^\bullet_T(F) \to S^\bullet(\mathfrak{t}^\vee)$ extends naturally to this localization, $H^\bullet_T(F)_{p_m} \to S^\bullet(\mathfrak{t}^\vee)_{p_m}$, and the integral on the RHS in Theorem 2.14 should be interpreted accordingly.

For the purpose of instanton counting a simpler localization formula suffices. Namely, the fixed point set will be discrete, eliminating the integral entirely. Furthermore, we want to perform localization with respect to the action of a non-compact torus $T_\mathbb{C}$, rather than of a compact torus as in Theorem 2.14. Suppose $T_\mathbb{C}$ acts holomorphically on a complex manifold M. It is easy to see that for a compact maximal torus $T \subset T_\mathbb{C}$, the set of $T_\mathbb{C}$-fixed points is identical to that of T-fixed points. Furthermore, complex-valued $T_\mathbb{C}$-equivariant (in the complex polynomial sense) forms can be identified with their restrictions to $\mathfrak{t} \subset \mathfrak{t}_\mathbb{C}$, turning them into complex valued T-equivariant forms (in the real polynomial sense). For these, Theorem 2.14 may be applied without restrictions, proving the corollary:

Corollary 2.15. *Let $T_\mathbb{C}$ be a non-compact torus acting holomorphically on a compact complex manifold M. If the fixed point set F is discrete, then for any equivariantly closed $\alpha \in \Omega^\bullet_{T_\mathbb{C},c}(M)$,*

$$\int_M \alpha = \sum_{f \in F} \frac{\iota_f^* \alpha}{\epsilon^{T_\mathbb{C}}(T_f M)}.$$

As all fixed points are isolated, the compactness of the manifold implies that the sum is finite.

To make sense of the quotients on the right, note that both $\iota_f^*\alpha$ and $\epsilon^{T_\mathbb{C}}(T_f M)$ (where $T_f M$ is the isotropy representation of $T_\mathbb{C}$ at f) are elements of $H^\bullet_{T_\mathbb{C}}(*)$ which we identify with $S^\bullet(\mathfrak{t}_\mathbb{C}^\vee)$. Under this identification, the equivariant Euler class corresponds to the product of the weights (multiset convention)

$$\epsilon^{T_\mathbb{C}}(E) = \prod_i w_i.$$

Thus, the sum in Corollary 2.15 is one of rational functions of $\mathfrak{t}_\mathbb{C}$, and the result is a polynomial.

More concretely, in Chapter 3 we will localize the integral

$$\int_{\mathfrak{M}(\underline{n},\underline{k})} \epsilon^{T_\mathbb{C}}(\mathbf{B})$$

of the equivariant Euler class of a $T_{\mathbb{C}}$-equivariant vector bundle over (a modified version of) the instanton moduli space. To be more precise, our application of the localization theorem to this integral will be merely formal as the moduli space $\mathfrak{M}(\underline{n}, \underline{k})$ is non-compact. Thus, the integral is defined as the sum over $T_{\mathbb{C}}$-fixed points

$$\sum_{\underline{\boldsymbol{\lambda}}} \frac{\iota_{\underline{\boldsymbol{\lambda}}}^* \epsilon^{T_{\mathbb{C}}}(\mathbf{B})}{\epsilon^{T_{\mathbb{C}}}(T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k}))}.$$

For the Euler class in the numerator, we have

$$\iota_{\underline{\boldsymbol{\lambda}}}^* \epsilon^{T_{\mathbb{C}}}(\mathbf{B}) = \epsilon^{T_{\mathbb{C}}}(\iota_{\underline{\boldsymbol{\lambda}}}^* \mathbf{B}) = \epsilon^{T_{\mathbb{C}}}(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}}),$$

where $\mathbf{B}|_{\underline{\boldsymbol{\lambda}}}$ is a $T_{\mathbb{C}}$-equivariant bundle over the fixed point $\underline{\boldsymbol{\lambda}}$, identified with the isotropy representation of $T_{\mathbb{C}}$. We proceed to work out this representation's weights and then $\epsilon^{T_{\mathbb{C}}}(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}})$ is just the product of these weights. In the denominator, $T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k})$ is equally identified with a $T_{\mathbb{C}}$-representation and we must again compute the product of its weights. This will be accomplished using some properties of $T_{\mathbb{C}}$-representations and their Euler classes. Finite-dimensional representations of $T_{\mathbb{C}}$ are completely reducible which implies that any short exact sequence of representations (where morphisms of representations are equivariant linear maps)

$$0 \to A \to B \to C \to 0$$

splits; $B \cong A + C$ (we use notation $+$ for direct sums of representations) and thus $\epsilon^{T_{\mathbb{C}}}(B) = \epsilon^{T_{\mathbb{C}}}(A)\epsilon^{T_{\mathbb{C}}}(C)$. This then implies the following lemma:

Lemma 2.16. *If*

$$\dots \xrightarrow{\alpha_{i-1}} A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \xrightarrow{\alpha_{i+2}} \dots$$

is a finite cochain complex of finite-dimensional complex $T_{\mathbb{C}}$-representations, then as virtual representations,

$$\sum_i (-1)^i A_i \cong \sum_i (-1)^i H_i,$$

where $H_i = \ker \alpha_i / \operatorname{im} \alpha_{i-1}$ are the quotient representations (the cohomology of the complex).

By virtual representation we mean a formal difference $V - W$ of $T_{\mathbb{C}}$-representations. Two virtual representations $V - W$, $V' - W'$ are defined to be isomorphic if and only if $V + W' \cong V' + W$. As we will outline in Section 2.6, the isomorphism types of virtual $T_{\mathbb{C}}$-representations form the $T_{\mathbb{C}}$-equivariant K-theory of the one-point space $*$.

Proof. There are short exact sequences

$$0 \to \ker \alpha_i \to A_i \to \operatorname{im} \alpha_i \to 0,$$
$$0 \to \operatorname{im} \alpha_{i-1} \to \ker \alpha_i \to H_i \to 0$$

of $T_{\mathbb{C}}$-representations. These split, so that $A_i \cong \ker \alpha_i + \operatorname{im} \alpha_i$ and $H_i \cong \ker \alpha_i - \operatorname{im} \alpha_{i-1}$. Thus

$$\sum_i (-1)^i A_i \cong \sum_i (-1)^i (\ker \alpha_i + \operatorname{im} \alpha_i) \cong \sum_i (-1)^i (\ker \alpha_i - \operatorname{im} \alpha_{i-1}) \cong \sum_i (-1)^i H_i.$$

□

The context where we will need this lemma is in calculating $\epsilon^{T_{\mathbb{C}}}(T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k}))$ as the product of the weights. To find the weights, we will realize $T_{\underline{\lambda}}\mathfrak{M}(\underline{n}, \underline{k})$ as the cohomology of a chain complex

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

of T-representations, with α injective and β surjective (so that the other cohomology representations vanish). The weights of the representations A, B, C will be apparent from their definitions, and the weights of $T_{\underline{\lambda}}\mathfrak{M}(\underline{n}, \underline{k})$ are found by expressing it as the virtual representation $-A + B - C$ using Lemma 2.16.

Localization for calculating symplectic volumes. One common application of equivariant localization is in calculating the volumes of compact symplectic manifolds (M, ω), that is, integrals

$$\int_M \frac{\omega^n}{n!} \tag{9}$$

where M is $2n$-dimensional. Suppose there is a Hamiltonian torus action $T \curvearrowright M$, i.e. a smooth action torus action that preserves the symplectic form ω and admits a moment map. By moment map we mean a map $\mu : M \to \mathfrak{t}^\vee$ satisfying $t_*\mu(X) = \mu(\mathrm{Ad}_t X)$ and $d\mu(X) = \iota_{\overline{X}}\omega$ for all $x \in M$, $t \in T$ and $X \in \mathfrak{t}$. Using μ, we can define an equivariant extension of the symplectic form:

$$\omega_{eq} \coloneqq \omega + \mu \in S^\bullet(\mathfrak{t}^\vee) \otimes \Omega^\bullet(M)$$

is a differential form-valued polynomial on the Lie algebra. We can check that it defines a T-equivariant form:

$$(\omega + \mu)(\mathrm{Ad}_t X) = \omega + t_*\mu(X) = t_*(\omega + \mu)(X)$$

Furthermore, this form is equivariantly closed:

$$D(\omega + \mu)(X) = d\omega - \iota_{\overline{X}}\omega + d\mu(X) - \iota_{\overline{X}}\mu(X) = d\mu(X) - \iota_{\overline{X}}\omega = 0$$

Then all powers $(\omega + \mu)^k$ and the exponential $\exp(\omega + \mu) = \sum_{k\geq 0} \frac{(\omega+\mu)^k}{k!}$ are also equivariantly closed and we can apply equivariant localization to the integral

$$\int_M \exp(\omega + \mu) = \sum_f \frac{\exp(\mu(f))}{\epsilon^T(T_f M)} \tag{10}$$

where the sum on the right is over all T-fixed points (we assume that the fixed point set is discrete). As an equivariant integral, this is an element of $H^\bullet_T(*) = S^\bullet(\mathfrak{t}^\vee)$, i.e. a polynomial on $\mathfrak{t}$, rather than a real number as (9). To recover the integral (9), we evaluate (10) on an element $X \in \mathfrak{t}$ of the Lie algebra and take the limit as $X \to 0$:

$$\int_M \frac{\omega^n}{n!} = \int_M \exp(\omega) = \lim_{X\to 0} \int_M \exp(\omega + \mu)(X) = \lim_{X\to 0} \sum_f \frac{\exp(\mu(f)X)}{\epsilon^T(T_f M)X}$$

In other words, we evaluate the rational function on the right at zero. This technique makes it possible to calculate symplectic volumes using equivariant localization.

We've essentially proved the Duistermaat–Heckman [DH82] formula:

Corollary 2.17 (Duistermaat–Heckman)**.** *If there is a Hamiltonian action of a compact torus T on a compact symplectic manifold (M, ω) and its fixed point set is finite, then for any moment map $\mu : M \to \mathfrak{t}^\vee$,*

$$\int_M \frac{\omega^n}{n!} e^\mu = \sum_{f \in M^T} \frac{e^{\mu(f)}}{\epsilon^T(T_f M)}.$$

It can be regarded as the original form of equivariant localization, generalized by Atiyah–Bott [AB84] and Berline–Vergne [BV82] to the non-symplectic setting as in Theorem 2.14.

2.6 Equivariant K-theory

Ordinary K-theory is a generalized cohomology theory of a space X defined by way of vector bundles over X. If X is a G-space, one may restrict attention to equivariant vector bundles, defining the equivariant K-theory of X. This has many commonalities with the equivariant cohomology, most importantly, there is a localization theorem. This allows instanton counting to be performed in the K-theory (rather than cohomology) setting, yielding a K-theoretic Nekrasov partition function $\mathcal{Z}^K_{\text{inst}}$ that is a sum over rational functions of exponential functions on the Lie algebra $\mathfrak{t}$, rather than a sum over rational functions (of linear functions) as in the cohomology case. Physically, this corresponds to five-dimensional gauge theories compactified on a circle [Nek03].

Throughout, all vector bundles are complex and have finite rank.

Definition 2.18. *Let X be a topological space and G a topological group acting continuously on X. The G-equivariant K-theory $K_G(X)$ of X is the Grothendieck group associated to the abelian monoid of complex G-equivariant vector bundles under direct sum.*

Elements of $K_G(X)$ are represented by formal differences $E \ominus F$ of G-equivariant complex vector bundles over X, and the equivalence relation on such formal differences is defined

$$E \ominus F \sim E' \ominus F' \iff E \oplus F' \oplus D \cong E' \oplus F \oplus D \tag{11}$$

for some G-equivariant vector bundle $D \to X$. These formal differences $E \ominus F$ are called virtual bundles and two equivalent virtual bundles are said to be isomorphic, written $E \ominus F \cong E' \ominus F'$ rather than with the symbol $\sim$. Note that in (11), the bundle D is the same on both sides. One could also define a reduced equivariant K-theory by instead defining $E \ominus F \sim E' \ominus F' \iff E \oplus F' \oplus D_1 \cong E' \oplus F \oplus D_2$ for two different bundles D_1, D_2 that arise as pullbacks of representations by the collapsing map $X \to *$. This has the disadvantage that the reduced equivariant K-theory of the one-point space $*$ is zero, which we do not want.

There are also higher equivariant K-groups $K^n_G(X)$ for $n > 0$, but we will not use those.

The Grothendieck group $K_G(X)$ is not just an abelian group under $\oplus$ but also a ring under tensor products $\otimes$. The similarities with equivariant cohomology don't end there. Like $H^\bullet_G(*)$, the equivariant K-theory of the one-point space is particularly simple: A G-equivariant vector bundle $V \to *$ may instead be regarded as a complex representation of G, facilitating an isomorphism

$$K_G(*) \cong R(G)$$

with the representation ring of G (which is defined as the Grothendieck group of the abelian monoid $\mathrm{Rep}(G)$ of finite-dimensional complex G-representations, in particular elements of $R(G)$ are isomorphism classes of virtual representations). For compact connected Lie groups and their complexifications, the representation character χ defines an isomorphism between the representation ring and the character ring. We only care about $T_{\mathbb{C}}$-representations, whose characters are

$$\chi \circ \exp = \sum_{w} m(w) e^{w}$$

where the sum is over the set of weights $w : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ lying in the weight lattice, and $m(w)$ is the weight's multiplicity. Each exponential is a Laurent monomial $e^{w} = t_1^{k_1} \dots t_n^{k_n}$ (where $(t_1, \dots, t_n) \in \mathbb{C}^{\times} \times \cdots \times \mathbb{C}^{\times} \cong T_{\mathbb{C}}$), identifying

$$K_{T_{\mathbb{C}}}(*) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

Equivariant maps $f : X \to Y$ between G-spaces permit pulling back equivariant bundles from Y to X. This extends to virtual bundles and descends to their equivalence classes, thus defining a pullback morphism $f^* : K_G(Y) \to K_G(X)$. Using the collapsing map $\pi : X \to *$, the pullback $\pi^* : R(G) = K_G(*) \to K_G(X)$ makes $K_G(X)$ into a module over the representation ring. G-Equivariant K-theory is then a contravariant functor from the category of G-spaces to that of $R(G)$-modules.

To discuss localization in equivariant K-theory, we must first define the K-theory equivariant Euler class of virtual equivariant vector bundles $E \to X$. This is different from the equivariant Euler class in cohomology and defined as

$$\epsilon^{G}(E) = \wedge_{-1} E^* = \sum_{k=0}^{\operatorname{rk} E} (-1)^k \wedge^k E^*,$$

an element of $K_G(X)$ (cf. (75) for an explanation of the notation $\wedge_{-1}$). It satisfies the usual properties of Euler classes $\epsilon^G(f^*E) = f^*\epsilon^G(E)$ and $\epsilon^G(E \oplus F) = \epsilon^G(E)\epsilon^G(F)$. For a sum of complex line bundles $E = L_1 \oplus \cdots \oplus L_n \to X$, it is

$$\epsilon^{G}(E) = \prod_{k} \epsilon^{G}(L_k) = \prod_{k} (1 - L_k^{-1}),$$

using $L^{-1} \cong L^*$.

In order to generalize the localization formula 2.15 to K-theory, we must first discuss what replaces the equivariant integral $\int_M \alpha$ in this context. In equivariant cohomology, the integral is a map $H_T^{\bullet}(M) \to H_T^{\bullet}(*) = S^{\bullet}(\mathfrak{t}^{\vee})$. In equivariant K-theory, we replace it by the pushforward map (Gysin homomorphism) $\pi_* : K_T(M) \to K_T(*)$ induced by the collapsing map $M \to *$. For general equivariant maps $f : M \to N$, pushforwards in equivariant K-theory are tricky to define and in general only exist if f is equivariantly K-orientable, see [Fok23, Sec. 4.2]. However, for the case where M is a compact complex manifold with holomorphic T-action and $\pi : M \to *$ is the collapsing map, they are guaranteed to exist.

Theorem 2.19. *Let M be a compact complex manifold with a holomorphic action by a compact or non-compact torus T, with isolated fixed points. Then, for any holomorphic T-equivariant vector bundle E over M,*

$$\pi_*(E) = \sum_{f \in M^T} \frac{\iota_f^* E}{\epsilon^T(T_f M)}.$$

The pushforward $\pi_*(E)$ is an element of the T-equivariant K-theory of the one-point space, which we identify with the representation ring of T, itself identified with the character ring $\mathbb{Z}[e^{\pm x_1}, \dots, e^{\pm x_r}]$

where $x_1, \ldots, x_r$ are coordinates on $\mathfrak{t}$ such that the lattice $\ker(\exp) \subset \mathfrak{t}$ consists of all vectors for which all x_i are integers.

For a complex representation V of a torus, we compute its K-theory Euler class using the weight space decomposition $V = \sum_w V_w$. Let us use the convention where the collection of weights is a multiset, i.e. weights with multiplicity are counted multiple times and the weight spaces are 1-dimensional. Then

$$\epsilon^T(V) = \epsilon^T\left(\sum_w V_w\right) = \prod_w \epsilon^T(V_w) = \prod_w (1 - e^{-w})$$

where w are the representation's weights. In the last step we identified the 1-dimensional representation $\epsilon^T(V_w) = 1 - V_w^*$ with its character $1 - e^{-w}$. In the convention where weights with multiplicity are counted only once, the formula is $\epsilon^T(V) = \prod_w (1 - e^{-w})^{m(w)}$ where $m(w)$ is the multiplicity.

Lemma 2.16 carries over to the K-theory Euler class, simplifying the computation of Euler classes of cohomology representations (like the tangent space of the instanton moduli space).

In instanton counting, the complex manifold will be the resolved instanton moduli space $\mathfrak{M}(\underline{n}, \underline{k})$. The bundle E whose pushforward we formally evaluate using Theorem 2.19 will be the virtual bundle $E = \epsilon^T(\mathbf{B})$. In this case $\iota_f^* \epsilon^T(\mathbf{B}) = \epsilon^T(\iota_f^* \mathbf{B})$ is the Euler class of a virtual representation which we define by

$$\epsilon^T(V - W) \coloneqq \frac{\epsilon^T(V)}{\epsilon^T(W)}$$

when $\epsilon^T(W) \neq 0$ (which will be the case for us). In the massless case or Section 3.2, where $\mathbf{B} = 0$, we will evaluate $\pi_*(1)$.

Last we mention that there's another generalized cohomology theory whose equivariant version has a localization theorem: Elliptic cohomology. A Nekrasov partition function can be examined in this theory too. In physics, it corresponds to six-dimensional theories. More details can be found in [Kim21].

3 Instanton counting

In Chapter 1, we defined the instanton partition function

$$\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathfrak{M}(n,k)} 1 \tag{12}$$

of massless $\mathcal{N} = 2$ supersymmetric $SU(n)$ or $U(n)$ gauge theory. This section is devoted to the instanton counting method of calculating this integral, introduced by Nekrasov in [Nek03]. The idea is to use equivariant localization as discussed in Section 2.5 to reduce the integral to a discrete sum over fixed points of the torus action on the instanton moduli space. These fixed points are parametrized by combinatorial objects called $\underline{n}$-colored partitions (cf. appendix A).

In this chapter, we start by defining basic gauge theory terms of instanton counting and defining framed instanton moduli spaces, their compactifications and resolutions based on the ADHM (Atiyah–Drinfeld–Hitchin–Manin) construction [Ati+78]. Following that, we perform rigorously the localization to fixed points of the integral (12), using a different method than those of Nekrasov [Nek03] and Nakajima–Yoshioka [NY05]. Namely, we are working purely within the ADHM linear data description of the moduli space. We proceed by introducing the $\mathcal{N} = 2$ supersymmetric quiver gauge theories, which have multi-factor gauge groups and matter fields, and generalize instanton counting to them.

Let us first, following [Kim21], provide some further context for the definition (12). To obtain (12), the Yang–Mills action is modified by adding the purely imaginary θ-term (θ is another coupling constant)

$$S_\theta[A] \coloneqq \frac{i\theta}{8\pi^2} \int_{\mathbb{R}^4} \operatorname{tr}(F_A \wedge F_A) = -i\theta k[A]$$

where $k[A]$ is the instanton charge we will discuss in Section 3.1. $S_\theta[A]$ is locally constant, meaning it does not change the Yang–Mills equations. In $\mathcal{N} = 2$ $U(n)$ or $SU(n)$ gauge theory, the full path integral $Z = \int [DA] e^{-S[A] - S_\theta[A]}$ localizes on the ASD instantons [Kim21, Sec. 1.3.2], producing the instanton partition function (12). The exponentiated coupling

$$\mathfrak{q} = e^{-\frac{8\pi^2}{g^2} + i\theta}$$

is given in terms of the coupling constant g of the Yang–Mills action (13) and the coupling constant θ of the θ-term. It may also be expressed as $\mathfrak{q} = e^{2\pi i \tau}$ in terms of the complexified coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$.

3.1 The instanton moduli space

Throughout this thesis, the spacetime is Wick-rotated (i.e. Euclidean) $\mathbb{R}^4$. As $\mathbb{R}^4$ is contractible, all principal G-bundles $P \to \mathbb{R}^4$ are trivial so that the choice of our principal bundle is immaterial. A gauge field on $\mathbb{R}^4$ is then a principal connection $A \in \Omega^1(P; \mathfrak{g})$. The Yang–Mills action of a connection A is defined as

$$S_{YM}[A] = \frac{1}{2g^2} \int_{\mathbb{R}^4} \operatorname{tr}(F_A \wedge *F_A), \tag{13}$$

where F_A is the connections curvature 2-form and $g > 0$ is a coupling constant. By Uhlenbeck's removable singularities theorem [Uhl82], if $P \to \mathbb{R}^4$ is a principal G-bundle with a finite-action connection

$A \in \Omega^1(P;\mathfrak{g})$, and G is compact, then P and A can be uniquely extended over the one-point compactification S^4. An instanton is a connection $A \in \Omega^1(P;\mathfrak{g})$ with finite Yang–Mills action which is anti-self-dual, meaning that for its curvature $*F_A = -F_A$, and which has finite Yang–Mills. Its charge is

$$k[A] := \frac{-1}{8\pi^2}\int_{S^4} \operatorname{tr}(F_A \wedge F_A),$$

(where we uniquely extend P and A to S^4) which is seen to be an integer as the second Chern class $c_2(P) \in H^4(S^4;\mathbb{Z})$ is identified with the de Rham cohomology class of $\frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A)$ in Chern–Weil theory [KN96]. The charge is non-positive for self-dual connections and non-negative for anti-self-dual connections [Lin11, Eq. 3.3.11].

For compact gauge group G, we define the framed moduli space of instantons

$$\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(G) := \{A \mid A \text{ a finite-action ASD connection on } P\}/\mathcal{G}_{\mathrm{small}} \tag{14}$$

where P is any principal G-bundle over $\mathbb{R}^4$ and $\mathcal{G}_{\mathrm{small}}$ consists of gauge transformations $g : \mathbb{R}^4 \to G$ that converge

$$g(x) \to 1 \qquad \text{as} \qquad \|x\| \to \infty$$

(''small'' gauge transformations). The subspace of $\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(G)$ consisting of instantons of charge k is denoted $\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(G,k)$.

Using the pullback from the universal bundle, principal G-bundles over S^4 are classified up to isomorphism by their class in $[S^4, BG] \cong \pi_4(BG) \cong \pi_3(G)$ (cf. [DK97]). In the cases $G = SU(n)$, $G = U(n)$, the integer $-k \in \mathbb{Z} \cong \pi_3(G)$ is the second Chern number $c_2(P) \in H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$. In cases where principal G-bundles over S^4 are classified by their second Chern class, we could also define $\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(G,k)$ as

$$\{(A,p) \mid A \text{ an ASD connection on } P, p \in P|_\infty \text{ a framing}\}/\mathcal{G} \tag{15}$$

where P is any principal G-bundle over S^4 with $c_2(P) = -k$, $\infty \in S^4 \setminus \mathbb{R}^4$ is the point at infinity, and $\mathcal{G}$ is the group of gauge transformations $g : S^4 \to G$ which acts on a framing $p \in P|_\infty$ by $g(\infty)^{-1}pg(\infty)$.

We first discuss $U(1)$-instantons. Isomorphism classes of principal $U(1)$-bundles over S^4 are classified by $\pi_3(U(1)) = 0$, meaning they are all trivial.

Lemma 3.1. *All $U(1)$-instantons on S^4 are flat, in the sense that their curvature $F_A = 0$. By Uhlenbeck's removable singularities theorem, the same result holds on $\mathbb{R}^4$ for finite-action instantons.*

Proof. As the principal $U(1)$-bundle $P \to S^4$ is trivial, there is a global gauge (by which we mean a section) $s : S^4 \to P$. An ASD connection A and may thus be regarded as a $\mathfrak{u}(1)$-valued 1-form on S^4. Its curvature $F = dA$ satisfies $*F = -F$, thus $d * F = -dF = 0$. Then

$$\Delta F = dd^*F + d^*dF = 0,$$

i.e. F is harmonic. By Hodge's theorem, on compact, oriented Riemannian manifolds, the vector space of harmonic p-forms is isomorphic to the p-th cohomology. Thus, as $H^2(S^4) = 0$, the curvature F must vanish. □

We see that $U(1)$ instantons on $\mathbb{R}^4$ are exactly the finite-action flat connections. These are all gauge-equivalent but not small gauge-equivalent, thus $\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(U(1))$ is non-trivial (more specifically, it is a circle).

In many places in the instanton counting literature one reads that the framed instanton moduli spaces for gauge groups $SU(n)$ and $U(n)$ are identical. This is not quite correct but the difference is inconsequential for the partition function.

Lemma 3.2. *The splitting* $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ *induces a bijection*

$$\mathfrak{M}^{\text{fr}}_{\text{inst}}(U(n), k) \cong \mathfrak{M}^{\text{fr}}_{\text{inst}}(SU(n), k) \times \mathfrak{M}^{\text{fr}}_{\text{inst}}(U(1), 0).$$

Proof. This follows from the splitting $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. Suppose $P \to \mathbb{R}^4$ is a principal $U(n)$-bundle. A gauge $s : \mathbb{R}^4 \to P$ identifies k-instantons with 1-forms $A \in \Omega^1(\mathbb{R}^4; \mathfrak{u}(n))$ such that

$$\frac{-1}{8\pi^2} \int \operatorname{tr}(F_A \wedge F_A) = k, \qquad *F_A = -F_A$$

where $F_A = dA + \frac{1}{2}[A \wedge A]$. The 1-form splits

$$A = A_{\mathfrak{su}(n)} + A_{\mathfrak{u}(1)}$$

where $A_{\mathfrak{su}(n)} \in \Omega^1(\mathbb{R}^4; \mathfrak{su}(n))$, $A_{\mathfrak{u}(1)} \in \Omega^1(\mathbb{R}^4; \mathfrak{u}(1))$ are themselves instantons (after introducing $SU(n)$ and $U(1)$ principal bundles and choosing gauges), and $k[A_{\mathfrak{su}(n)}] = k$, $k[A_{\mathfrak{u}(1)}] = 0$. That this splitting descends to the framed moduli spaces follows from the unique decomposition

$$g = g_{SU(n)} g_{U(n)}$$

of any (small) gauge transformation $g : \mathbb{R}^4 \to U(n)$ into (small) gauge transformations $g_{SU(n)} : \mathbb{R}^4 \to SU(n)$ and $g_{U(1)} : \mathbb{R}^4 \to U(1)$. □

While the framed moduli spaces of $SU(n)$ and $U(n)$ differ, in the literature $\mathcal{Z}_{\text{inst}} = \sum_{k \geq 0} \mathfrak{q}^k \int_{\mathfrak{M}(n,k)} 1$ still serves as the instanton partition function's definition for both gauge theories (at least in the massless case). The reason is likely that their unframed moduli spaces really are identical, as that of $U(1)$ is trivial, and that the path-integral can be split

$$\begin{aligned} \int [DA_{\mathfrak{u}(n)}] \mathfrak{q}^{k[A_{\mathfrak{u}(n)}]} &= \int [DA_{\mathfrak{u}(1)}] \mathfrak{q}^{k[A_{\mathfrak{u}(1)}]} \int [DA_{\mathfrak{su}(n)}] \mathfrak{q}^{k[A_{\mathfrak{su}(n)}]} \\ &= \int [DA_{\mathfrak{u}(1)}] \int [DA_{\mathfrak{su}(n)}] \mathfrak{q}^{k[A_{\mathfrak{su}(n)}]}. \end{aligned}$$

Last, we mention that $\mathfrak{M}^{\text{fr}}_{\text{inst}}(SU(n), k)$ is a hyperkähler manifold. The construction (14) is a hyperkähler quotient where F_A^+ (the self-dual part of the curvature) plays the role of the moment map [Nak99, below Thm. 3.46].

The ADHM construction [Ati+78]. For the vector spaces $N = \mathbb{C}^n$, $K = \mathbb{C}^k$, we define the vector space

$$X := \operatorname{Hom}(K, K) \oplus \operatorname{Hom}(K, K) \oplus \operatorname{Hom}(N, K) \oplus \operatorname{Hom}(K, N)$$

and denote elements, called linear ADHM data, by $(B_1, B_2, I, J) \in X$.

This space X may be regarded as the cotangent bundle $T^*(\operatorname{Hom}(K, K) \oplus \operatorname{Hom}(N, K))$ giving it a natural symplectic structure. We define two complex structures on the vector space X; multiplication by

i and

$$j : (B_1, B_2, I, J) \mapsto (B_2^\dagger, -B_1^\dagger, J^\dagger, -I^\dagger).$$

As j is anti-linear ($ji = -ij$) and the complex structures are compatible with the symplectic structure, X is a hyperkähler manifold.

We have a $GL(k, \mathbb{C})$-action on X given by

$$g \cdot (B_1, B_2, I, J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \tag{16}$$

for $g \in GL(k, \mathbb{C})$. Its infinitesimal version $\mathfrak{gl}(k, \mathbb{C}) \to T_{(B_{1,2}, I, J)}X$ at $(B_{1,2}, I, J) \in X$ is

$$\phi \mapsto ([\phi, B_1], [\phi, B_2], \phi I, -J\phi). \tag{17}$$

When the action (16) is restricted to the compact subgroup $U(k) \subset GL(k, \mathbb{C})$, it preserves the hyperkähler structure on X: Denoting by $\alpha : \mathfrak{u}(k) \to T_{(B_{1,2}, I, J)}X$ the restricted infinitesimal action, $\alpha i = i\alpha$ and $\alpha j = j\alpha$ can easily be checked, showing that the $U(k)$-action on X preserves the complex structures. The full action by $GL(k, \mathbb{C})$ does not preserve the hyperkähler structure. Clearly the symplectic structure induced by regarding X as a cotangent bundle is also preserved under $U(k)$.

We define maps $\mu^r, \mu^c : X \to \mathrm{Hom}(K, K)$,

$$\mu^r(B_{1,2}, I, J) = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger + J^\dagger J$$
$$\mu^c(B_{1,2}, I, J) = [B_1, B_2] + IJ$$

where μ^r is valued in Hermitian matrices. Using the isomorphism $\mathfrak{u}(k) \cong \mathfrak{u}(k)^\vee$ induced by the standard Hermitian product, these maps are related to maps $\mu_{1,2,3} : X \to \mathfrak{u}(k)^\vee$ by

$$\mu_1 = \frac{\mathrm{i}}{2}\mu^r,$$
$$\mu^c = \mu_2 + i\mu_3,$$

and $\vec{\mu} = (\mu_1, \mu_2, \mu_3) : M \to \mathbb{R}^3 \otimes \mathfrak{u}(k)^\vee$ is a hyperkähler moment map for the $U(k)$-action (16). This means that $\mu_i(gx) = \mathrm{Ad}^*_{g^{-1}}\mu_i(x)$ for all $g \in U(k)$ and $d\mu_i(v)\xi = \omega_i(\bar{\xi}, v)$ for all $v \in TX$ and $\xi \in \mathfrak{u}(k)$ where $\bar{\xi}$ is the vector field generated by ξ through the infinitesimal $U(k)$-action and ω_i are the three symplectic forms.

In [Hit+87], Hitchin et al. proved a quotient construction for hyperkähler manifolds from existing hyperkähler manifolds X and their moment maps.

Theorem 3.3. *Let X be a hyperkähler manifold with a compact Lie group G acting on X in a way that preserves the hyperkähler structure. Let $\vec{\mu} : X \to \mathbb{R}^3 \otimes \mathfrak{g}^\vee$ be a hyperkähler moment map and suppose that $\vec{\zeta} \in \mathbb{R}^3 \otimes \mathfrak{g}^\vee$ satisfies that $\mathrm{Ad}^*_g\vec{\zeta} = \vec{\zeta}$ for all $g \in G$ and that the G-action on $\vec{\mu}^{-1}(\vec{\zeta})$ is free. Then the quotient space $\vec{\mu}^{-1}(\vec{\zeta})/G$ is a smooth manifold with a hyperkähler structure induced by that on X.*

By the famous ADHM construction [Ati+78], data $(B_{1,2}, I, J) \in X$ for which μ^r and μ^c vanish can be used to construct an $SU(n)$-instanton of charge k on $\mathbb{R}^4$. We shall not give any details of the ADHM construction. In any case, by Theorem 3.3,

$$\mathfrak{M}_0^{\mathrm{reg}}(n, k) := \{(B_{1,2}, I, J) \in X \mid \mu^r(B_{1,2}, I, J) = \mu^c(B_{1,2}, I, J) = 0$$

and the stabilizer $U(k)_{(B_{1,2},I,J)}$ is trivial$\}/U(k)$

is a hyperkähler quotient (as the trivial stabilizer condition is open). By the ADHM construction it is isomorphic (as hyperkähler manifolds) to the moduli space $\mathfrak{M}^{\mathrm{fr}}_{\mathrm{inst}}(SU(n),k)$ of framed k-instantons for gauge group $SU(n)$. While this space is non-singular, it is also non-compact. There is the small instanton singularity: Clearly, if $(B_{1,2},I,J) \in X$ then so is $\rho(B_{1,2},I,J)$ for any $\rho \in \mathbb{C}^\times$ but the limit for $\rho = 0$ has non-trivial stabilizer, hence it is not contained in the moduli space.

This non-compactness was overcome by Uhlenbeck [Uhl82], introducing point-like instantons to the moduli space:

$$\mathfrak{M}_0(n,k) \coloneqq \mathfrak{M}_0^{\mathrm{reg}}(n,k) \cup \bigcup_{k'=0}^{k-1} \mathfrak{M}_0^{\mathrm{reg}}(n,k') \times s^{k-k'}\mathbb{R}^4$$

where $s^{k-k'}\mathbb{R}^4$ consists of multisets of cardinality $k-k'$ in $\mathbb{R}^4$. Each point in a multiset corresponds to a pointlike instanton concentrated at that point. Pairs $([A],(x_1,\dots,x_l))$ are called ideal instantons and $\mathfrak{M}_0(n,k)$ is the framed moduli space of ideal instantons. This is a compactification of $\mathfrak{M}_0^{\mathrm{reg}}(n,k)$ but it has singularities.

Resolving the singularities in $\mathfrak{M}_0(n,k)$. Adding ideal instantons to the moduli space introduces singularities. We define a new moduli space (by two different definitions) that resolves these.

Definition 3.4. *We say that ADHM data $(B_{1,2},I,J) \in X$ are stable if they satisfy the following stability condition: Any subspace $S \subseteq K$ that contains* im I *and is invariant under B_1 and B_2 must in fact be equal to K.*

We say that $(B_{1,2},I,J)$ are co-stable if any $B_{1,2}$-invariant subspace $S \subseteq N$ contained in ker J *is zero.*

We will only work with the stability condition for our moduli space. Rephrasing the stability condition as saying that K has a basis consisting of vectors B^*I_α where B^* is some product of $B_{1,2}$ and I_α is the α-th column of I demonstrates that the stability condition is open; small perturbations of $(B_{1,2},I,J)$ preserve the linear independence of the basis vectors B^*I_α.

We prove now

Lemma 3.5. *Suppose $(B_{1,2},I,J)$ satisfy the stability condition. Then the stabilizer of the $GL(k,\mathbb{C})$-action on $(B_{1,2},I,J)$ is trivial.*

Proof. Suppose for $g \in GL(k,\mathbb{C})$ that $(gB_{1,2}g^{-1}, gI, Jg^{-1}) = (B_{1,2},I,J)$. Then $gB_{1,2} = B_{1,2}g$ and $(g-\mathbb{1})I = 0$, i.e. $\operatorname{im} I \subseteq \ker(g-\mathbb{1})$. Now, if $v \in \ker(g-\mathbb{1})$, then $(g-\mathbb{1})B_{1,2}v = B_{1,2}(g-\mathbb{1})v = 0$, so $\ker(g-\mathbb{1})$ is closed under multiplication by $B_{1,2}$. Thus, by the stability condition, $\ker(g-\mathbb{1}) = K$, i.e. $g = \mathbb{1}$. □

Lemma 3.6. *If $(B_{1,2},I,J) \in X$ has trivial stabilizer in $GL(k,\mathbb{C})$, then $d\mu^c_{(B_{1,2},I,J)}$ has full rank.*

Proof. At $(B_{1,2},I,J)$, the derivative is

$$d\mu^c_{(B_{1,2},I,J)} : (\delta B_{1,2}, \delta I, \delta J) \mapsto [\delta B_1, B_2] + [B_1, \delta B_2] + (\delta I)J + I(\delta J).$$

The cokernel of this is given by matrices $\phi \in \mathbb{C}^{k\times k}$ which are orthogonal to the image, i.e.

$$\mathrm{coker} = \{\phi \in \mathbb{C}^{k\times k} \mid \mathrm{tr}\left(\phi^\dagger d\mu^c_{(B_{1,2},I,J)}(\delta B_{1,2},\delta I,\delta J)\right) = 0 \text{ for any tangent } (\delta B_{1,2},\delta I,\delta J)\}$$

$$= \{\phi \in \mathbb{C}^{k\times k} \mid [\phi^\dagger, B_1] = [\phi^\dagger, B_2] = 0, \phi^\dagger I = 0, J\phi^\dagger = 0\}.$$

But, comparing with (17) this means that the infinitesimal action of $\phi^\dagger \in \mathfrak{gl}(k,\mathbb{C})$ at $(B_{1,2}, I, J)$ is zero. As the stabilizer of $(B_{1,2}, I, J)$ in $GL(k,\mathbb{C})$ is trivial, it must then be that $\phi = 0$. □

Thus we find that $d\mu^c$ has full rank at all stable points. Together with Lemma 3.5, this shows that $\{\mu^c = 0, \text{stability condition}\}$ is a smooth complex variety. As both $\mu^c = 0$ and the stability condition are $GL(k,\mathbb{C})$-invariant, we can quotient by the $GL(k,\mathbb{C})$-action. By Lemma 3.5, this quotient is non-singular, i.e. a complex manifold, of dimension $2nk$:

$$\mathfrak{M}(n,k) := \{\mu^c = 0 \text{ and the stability condition (def. 3.4) holds}\}/GL(k,\mathbb{C}) \tag{18}$$

This is the resolved moduli space we will work with in our treatment of instanton counting. Note that it is not compact: Rescaling $[B_{1,2}, I, J] \in \mathfrak{M}(n,k)$ by $\rho \in \mathbb{C}^\times$ preserves both $\mu^c = 0$ and the stability condition, but if $\rho \to 0$ or $\rho \to \infty$, $[\rho B_{1,2}, \rho I, \rho J]$ doesn't converge in $\mathfrak{M}(n,k)$.

As shown in [Bar77], the space $\mathfrak{M}(n,k)$ can be interpreted as the framed moduli space of torsion-free sheaves on $\mathbb{P}^2$ of rank k and second Chern number n. A framed torsion-free sheaf is a pair (E,Φ) where E is a torsion-free sheaf on $\mathbb{P}^2$ with $\operatorname{rk} E = k$ and $c_2(E) = n$ (using $H^4(\mathbb{P}^2;\mathbb{Z}) \cong \mathbb{Z}$) which is locally free in a neighborhood of $\ell_\infty \subset \mathbb{P}^2$, the line at infinity. The framing at infinity is an isomorphism $\Phi : E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus k}$. In the simple case where $k = 1$, $\mathfrak{M}(n,1)$ is isomorphic to the Hilbert scheme of n points on $\mathbb{C}^2$. This interpretation and the necessary geometric invariant theory background is a main focus of Nakajima's lecture notes [Nak99].

In fact, the framed moduli space of instantons $\mathfrak{M}_0^{\text{reg}}(n,k)$ has a similar algebro-geometric interpretation, identifying it with the framed moduli space of locally free sheaves on $\mathbb{P}^2$, as shown by Donaldson in [Don84].

The space $\mathfrak{M}(n,k)$ can be equipped with a hyperkähler structure. To do this, one has to prove that it is diffeomorphic to the hyperkähler quotient

$$\mathfrak{M}_\zeta(n,k) := \{(\mu^r, \mu^c) = (\zeta \mathbb{1}_K, 0)\}/U(k)$$

where $\zeta > 0$ is an arbitrary real constant. The isomorphism type of $\mathfrak{M}_\zeta(n,k)$ is independent of ζ so long as $\zeta \neq 0$ but, as we defined $\mathfrak{M}(n,k)$ using the stability condition, the isomorphism with $\mathfrak{M}(n,k)$ is easier to construct in the positive case. For the negative case, one would instead work with the co-stability condition in the definition (18).

Lemma 3.7. *If $(B_{1,2}, I, J) \in X$ satisfy the deformed real ADHM equation $\mu^r = \zeta > 0$, then we have the following stability condition: Any subspace $S \subseteq K$ that contains* $\operatorname{im} I$ *and is invariant under B_1 and B_2 must in fact be equal to K.*

If instead $\mu^r = \zeta < 0$, then we have the co-stability condition that any $B_{1,2}$-invariant subspace $S \subseteq N$ contained in $\ker J$ *is zero.*

Proof. We prove the case $\zeta > 0$, the negative case is analogous. To be shown is that for a vector space $S \supseteq \operatorname{im} I$ invariant under $B_{1,2}$, its orthogonal complement $S^\perp \subseteq K$ is trivial. Define the orthogonal projection $P : K \to S^\perp$. Furthermore, $PB_{1,2}(1-P) = 0$, $(1-P)^2 = 1-P$ and $P^\dagger = P$, implying

$$\operatorname{tr}\left[PB_{1,2}B_{1,2}^\dagger P - PB_{1,2}^\dagger B_{1,2}P\right] = \operatorname{tr}\left[PB_{1,2}B_{1,2}^\dagger P - PB_{1,2}^\dagger B_{1,2}P + PB_{1,2}^\dagger PB_{1,2}P - PB_{1,2}PB_{1,2}^\dagger P\right]$$

$$= \operatorname{tr}\left[PB_{1,2}(1-P)B_{1,2}^\dagger P - PB_{1,2}^\dagger(1-P)B_{1,2}P\right]$$
$$= -\operatorname{tr}\left[((1-P)B_{1,2}P)^\dagger(1-P)B_{1,2}P\right] \le 0.$$

Using the ADHM equation $\mu^r = \zeta \mathbb{1}_K$, we have

$$\zeta \mathbb{1}_{S^\perp} = P\mu^r P^\dagger = P\left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\right)P^\dagger$$
$$= P[B_1, B_1^\dagger]P + P[B_2, B_2^\dagger]P - (PJ^\dagger)(PJ^\dagger)^\dagger$$

where we used that $PI = 0$. Taking the trace, we get

$$0 \le \operatorname{tr} \zeta \mathbb{1}_{S^\perp} = -\operatorname{tr}(PJ^\dagger)(PJ^\dagger)^\dagger - \operatorname{tr}\left[((1-P)B_{1,2}P)^\dagger(1-P)B_{1,2}P\right] \le 0.$$

Finally, $\operatorname{tr} \mathbb{1}_{S^\perp} = 0$ implies that $S^\perp = 0$. □

This lemma, together with Lemma 3.5, shows that the quotient $\mathfrak{M}_\zeta(n,k)$ is non-singular. Furthermore, the lemma shows that the obvious map

$$f : \{\mu^r = \zeta, \mu^c = 0\}/U(k) = \mathfrak{M}_\zeta(n,k) \to \mathfrak{M}(n,k) = \{\mu^c = 0, \text{ stability}\}/GL(k,\mathbb{C}),$$

sending $[B_{1,2}, I, J]_{U(k)}$ to $[B_{1,2}, I, J]_{GL(k,\mathbb{C})}$ is well-defined. That this map is an isomorphism of complex manifold is a special case of a theorem for quiver varieties: These have two different constructions generalizing $\mathfrak{M}(n,k)$ and $\mathfrak{M}_\zeta(n,k)$ and these constructions are equivalent [Kir16, Theorem 10.49]. Our moduli spaces $\mathfrak{M}(n,k)$, $\mathfrak{M}_\zeta(n,k)$ are the Jordan ($\hat{A}_0$) quiver's quiver variety.

Using standard tools relating hyperkähler quotients and algebro-geometric quotients (as presented in [Nak99]), the framed moduli space of ideal instantons $\mathfrak{M}_0(n,k)$ can be shown to be in bijective correspondence with the affine algebro-geometric quotient $\{\mu^c = 0\}//GL(k,\mathbb{C})$ (cf. [Nak94b, Proposition 2.1]). Then there's a natural map $\pi : \mathfrak{M}_\zeta(n,k) \to \mathfrak{M}_0(n,k)$ and Nakajima [Nak94b, Theorem 2.2] proved that it is a resolution of the singularities introduced by Uhlenbeck compactification.

Interpretation on non-commutative $\mathbb{R}^4$. In [NS98], the resolved moduli space $\{\mu^r = \zeta, \mu^c = 0\}/U(k)$ was shown by Nekrasov and Schwarz to be the moduli space of $U(n)$ k-instantons on non-commutative $\mathbb{R}^4 = \langle x_1, x_2, x_3, x_4\rangle$ with relations

$$[z_0, \bar{z}_0] = [z_1, \bar{z}_1] = -\frac{\zeta}{2}$$

where $z_0 = x_1 + ix_2$, $z_1 = x_3 + ix_4$. On non-commutative $\mathbb{R}^4$ not all $U(1)$-instantons are gauge-equivalent, unlike for commutative $\mathbb{R}^4$.

3.2 Fixed points and equivariant localization

In [Nek03], Nekrasov transformed the equivariant integral (12) defining the instanton partition function as a sum over n-colored partitions of size k (cf. appendix A). He applied a localization method introduced in [MNS00] to reduce the integral to a multi-dimensional contour integral. The sum then arises by applying the residue formula to this contour integral.

We will transform the integral (12) using a different localization procedure, applying Theorem 2.14. To do this, we introduce a torus action on $\mathfrak{M}(n,k)$, locate all its fixed points, and compute the Euler classes of the tangent isotropy representations. Nakajima and Yoshioka [NY05] found the fixed points using the identification of $\mathfrak{M}(n,k)$ with the framed moduli space of torsion-free sheaves on $\mathbb{P}^2$, mentioned in the previous section. Our process differs from theirs therein that we make no use of this interpretation of the moduli space, but rather work solely within the ADHM description (18). After locating the fixed points, we must calculate the Euler classes of the tangent isotropy representations. The partition function is then the sum of their reciprocals.

3.2.1 Fixed points

In the previous subsection we discussed the $GL(k,\mathbb{C})$-action

$$g \in GL(k,\mathbb{C}) : \qquad g\cdot(B_1,B_2,I,J) = (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1})$$

on X, by which we quotiented to construct $\mathfrak{M}(n,k)$. We introduce holomorphic actions by $GL(n,\mathbb{C})$ (the complexified gauge group, acting on the right) and $(\mathbb{C}^\times)^2$ (the complexified maximal torus of Spin(4)) on X which commute with the $GL(k,\mathbb{C})$-action and thus descend to the quotient $\mathfrak{M}(n,k)$.

$$h \in GL(n,\mathbb{C}) : \qquad (B_1,B_2,I,J)\cdot h \coloneqq (B_1,B_2,Ih,h^{-1}J) \tag{19}$$

$$(q_1,q_2) \in (\mathbb{C}^\times)^2 : \qquad (q_1,q_2)\cdot(B_1,B_2,I,J) \coloneqq (q_1B_1,q_2B_2,I,qJ), \tag{20}$$

where we define $q \coloneqq q_1q_2$. The corresponding infinitesimal actions $\mathfrak{gl}(k,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}), \mathbb{C}^2 \to T_{(B_{1,2},I,J)}X$ at (B_1,B_2,I,J) are:

$$\phi \in \mathfrak{gl}(k,\mathbb{C}) : \qquad \phi \mapsto ([B_1,\phi],[B_2,\phi],-\phi I, J\phi) \tag{21}$$

$$\mathfrak{a} \in \mathfrak{gl}(n,\mathbb{C}) : \qquad \mathfrak{a} \mapsto (B_1,B_2,I\mathfrak{a},-\mathfrak{a}J) \tag{22}$$

$$\vec{\varepsilon} = (\varepsilon_1,\varepsilon_2) \in \mathbb{C}\oplus\mathbb{C} : \qquad \vec{\varepsilon} \mapsto (\varepsilon_1B_1,\varepsilon_2B_2,I,\varepsilon J), \tag{23}$$

where we define $\varepsilon \coloneqq \varepsilon_1+\varepsilon_2$.

We restrict the action of $GL(n,\mathbb{C})\times(\mathbb{C}^\times)^2$ to T, the complexification of the maximal torus in $U(n)\times(\mathbb{C}^\times)^2$. This is a non-compact torus, isomorphic to $(\mathbb{C}^\times)^{n+2}$. Then an element $(\mathfrak{a},\vec{\varepsilon}) \in \mathfrak{t}$ consists of a complex diagonal matrix and a pair of complex numbers:

$$\mathfrak{a} = \begin{pmatrix} \mathfrak{a}_1 & & & \\ & \mathfrak{a}_2 & & \\ & & \ddots & \\ & & & \mathfrak{a}_n \end{pmatrix} \qquad \text{and} \qquad \vec{\varepsilon} = (\varepsilon_1,\varepsilon_2).$$

Note that here, unlike in Chapter 2, we denote the non-compact torus by T rather than $T_{\mathbb{C}}$. While the $GL(n,\mathbb{C})$-action is on the right, the action by T is on the left as T is abelian.

Now let us find the fixed points of the action of T on $\mathfrak{M}(n,k)$. For a point $(B_{1,2},I,J)\in X$ to represent a T-fixed point in the quotient $\mathfrak{M}(n,k) = \{\mu^c = 0,\ \text{stability}\}/GL(k,\mathbb{C})$, it must be that the action of T preserves the $GL(k,\mathbb{C})$-orbit of $(B_{1,2},I,J)$. That is, for every $(e^{\mathfrak{a}},e^{\vec{\varepsilon}}) \in T$ there must be a $e^{\phi} \in GL(k,\mathbb{C})$ such that

$$(e^{\mathfrak{a}},e^{\vec{\varepsilon}})\cdot(B_{1,2},I,J) = e^{\phi}\cdot(B_{1,2},I,J) \tag{24}$$

and this is unique by the freeness of the $GL(k,\mathbb{C})$-action. It is easy to check (cf. Lemma 3.12) that

$(e^{\mathfrak{a}}, e^{\vec{\varepsilon}}) \mapsto e^{\phi}$ is a homomorphism of Lie groups and

$$\phi : \mathfrak{t} \longrightarrow \mathfrak{gl}(k, \mathbb{C}), \quad (\mathfrak{a}, \vec{\varepsilon}) \mapsto \phi(\mathfrak{a}, \vec{\varepsilon})$$

a homomorphism of Lie algebras. We will mostly suppress the variables $\mathfrak{a}, \vec{\varepsilon}$ in the function and just write ϕ for $\phi(\mathfrak{a}, \vec{\varepsilon})$.

The infinitesimal version of equation (24) is that (21) = (22) + (23), i.e.

$$[\phi, B_{1,2}] = \varepsilon_{1,2} B_{1,2}, \tag{25}$$

$$\phi I = I\mathfrak{a}, \tag{26}$$

$$J\phi = \mathfrak{a}J - \varepsilon J. \tag{27}$$

These equations put constraints on the homomorphism ϕ. We call them the fixed point equations.

We denote by $I_\alpha \in \mathbb{C}^{k\times 1}$ the columns of matrix I and by $J_\alpha \in \mathbb{C}^{1\times k}$ the rows of matrix J:

$$I = \begin{pmatrix} I_1 & \dots & I_n \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} J_1 \\ \vdots \\ J_n \end{pmatrix}.$$

For us α is always an index ranging from 1 to n. Using these matrix decompositions, equations (26), (27) can be written as

$$\phi I_\alpha = \mathfrak{a}_\alpha I_\alpha \tag{28}$$

$$J_\alpha \phi = (\mathfrak{a}_\alpha - \varepsilon) J_\alpha \tag{29}$$

meaning that I_α and J_α are respectively left or right eigenvectors of ϕ. In fact, the following lemma shows that we have even more information about the spectral properties of ϕ.

Lemma 3.8. *Suppose $(B_{1,2}, I, J) \in X$ and $\phi : \mathfrak{t} \longrightarrow \mathfrak{gl}(k, \mathbb{C})$ satisfy the fixed point equations (25), (26), (27). Then for any product B^* of the matrices $B_{1,2}$ with f_1 factors of B_1 and f_2 factors of B_2 (i.e. some reordering of $B_1^{f_1} B_2^{f_2}$), we have*

$$\phi B^* I_\alpha = (\mathfrak{a}_\alpha + f_1 \varepsilon_1 + f_2 \varepsilon_2) B^* I_\alpha$$

and

$$J_\alpha B^* \phi = (\mathfrak{a}_\alpha - (f_1 + 1)\varepsilon_1 - (f_2 + 1)\varepsilon_2) J_\alpha B^*.$$

Proof. The proof goes by induction. The base case where B^* is the empty product is just (28), (29). Now, if the lemma holds for B^* a product of f_1 factors B_1 and f_2 factors B_2, then, using the commutation relations (25) and the induction hypothesis,

$$\begin{aligned} \phi B_{1,2} B^* I_\alpha &= \left(B_{1,2}\phi + \varepsilon_{1,2} B_{1,2}\right) B^* I_\alpha \\ &= B_{1,2} \left(\mathfrak{a}_\alpha + f_1\varepsilon_1 + f_2\varepsilon_2 + \varepsilon_{1,2}\right) B^* I_\alpha \end{aligned}$$

and

$$J_\alpha B^* B_{1,2} \phi = J_\alpha B^* \left(\phi B_{1,2} - \varepsilon_{1,2} B_{1,2}\right)$$

$$= J_\alpha B^* \left(\mathfrak{a}_\alpha - (f_1+1)\varepsilon_1 - (f_2+1)\varepsilon_2 - \varepsilon_{1,2}\right) B_{1,2}$$

which is what was to be proved. □

Using this we can show that $J = 0$: The stability condition from definition (3.4) is equivalent to the vector space K being generated by vectors $B^* I_\beta$ where B^* is some product of f_1 factors of B_1 and f_2 factors of B_2 and $\beta = 1, \dots, n$. By Lemma 3.8, for any $\alpha, \beta = 1, \dots, n$, we have

$$(\mathfrak{a}_\alpha - \varepsilon_1 - \varepsilon_2) J_\alpha B^* I_\beta = J_\alpha \phi B^* I_\beta = (\mathfrak{a}_\beta + f_1\varepsilon_1 + f_2\varepsilon_2) J_\alpha B^* I_\beta.$$

We can easily choose $\mathfrak{a}$ and $\vec{\varepsilon}$ such that, for all $f_1, f_2 \in \mathbb{Z}_{\geq 0}$,

$$\mathfrak{a}_\alpha - \varepsilon_1 - \varepsilon_2 \neq \mathfrak{a}_\beta + f_1\varepsilon_1 + f_2\varepsilon_2.$$

Hence it must be that $J_\alpha(B^* I_\beta) = 0$ and thus, as vectors $B^* I_\beta$ generate K, the linear map J must be zero. Essentially we argue that, as the left and right eigenvalues of ϕ are identical but the sets of left and right eigenvalues suggested by Lemma 3.8 are disjoint, the purported right eigenvectors J_α must vanish.

Now the complex ADHM equation $\mu^c(B_{1,2}, I, J) = 0$ together with $J = 0$ imply that $[B_1, B_2] = 0$. Thus we have proved

Lemma 3.9. *Suppose $(B_{1,2}, I, J) \in X$ represents a T-fixed point in $\mathfrak{M}(n,k)$. Then $J = 0$ and $B_{1,2}$ commute.*

We wish to identify the fixed point represented by $(B_{1,2}, I, J)$ with an n-colored partition $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^n)$. Now that we know $B_{1,2}$ commute, the stability condition (def. (3.4)) implies that the set

$$\mathcal{G} \coloneqq \{b_{\alpha,(s_1,s_2)} \coloneqq B_1^{s_1-1} B_2^{s_2-1} I_\alpha \mid s_1, s_2 = 1, 2, \dots \text{ and } \alpha = 1, \dots, n\}$$

generates the vector space K. Lemma 3.8 tells us that $b_{\alpha,(s_1,s_2)}$ is an eigenvector of ϕ with eigenvalue

$$\phi_{\alpha,(s_1,s_2)} \coloneqq \mathfrak{a}_\alpha + (s_1 - 1)\varepsilon_1 + (s_2 - 1)\varepsilon_2.$$

We can choose $\mathfrak{a}$ and $\vec{\varepsilon}$ such that all numbers $\phi_{\alpha,(s_1,s_2)}$ are distinct, meaning all eigenvalues of ϕ have multiplicity 1. Of course, K is of finite dimension k, thus all but k of the vectors $b_{\alpha,(s_1-1,s_2-1)}$ must vanish. If $b_{\alpha,(s_1-1,s_2-1)} = 0$, then

$$b_{\alpha,(s_1,s_2-1)} = B_1 b_{\alpha,(s_1-1,s_2-1)} = 0 \quad \text{and} \quad b_{\alpha,(s_1-1,s_2)} = B_2 b_{\alpha,(s_1-1,s_2-1)} = 0$$

must also be zero. This means that for each $\alpha = 1, \dots, n$, the combinations $(s_1, s_2) \in \mathbb{Z}^2_{>0}$ for which $b_{\alpha,(s_1-1,s_2-1)}$ is non-zero, must form a Young diagram (cf. appendix A), with corresponding partition

$$\lambda^\alpha = (\lambda^\alpha_1 \geq \lambda^\alpha_2 \geq \dots), \qquad \lambda^\alpha_{s_1} = \max\{s_2 \in \mathbb{Z}_{>0} \mid B_1^{s_1-1} B_2^{s_2-1} I_\alpha \neq 0\},$$

where we define $\max \emptyset \coloneqq 0$. Summing up:

Lemma 3.10. *Suppose $(B_{1,2}, I, J) \in X$ represents a T-fixed point of $\mathfrak{M}(n,k)$. Then there exists a unique n-colored partition $\boldsymbol{\lambda}$, determined by the fixed point, such that*

$$\mathcal{B} \coloneqq \{b_{\alpha\square} \coloneqq B_1^{s_1-1} B_2^{s_2-1} I_\alpha \mid \alpha = 1, \dots, n \text{ and } \square = (s_1, s_2) \in \lambda^\alpha\}$$

is a basis of K.

Proof. The preceding discussion showed that such an n-colored partition λ exists and is uniquely determined by $(B_{1,2}, I, J)$. That the partition is independent of the particular representative of the fixed point is easy to check; any such representative will be related to $(B_{1,2}, I, J)$ by the action of a matrix $A \in GL(k, \mathbb{C})$ and the resulting basis will be $b'_{\alpha\square} = Ab_{\alpha\square}$. $\square$

Lastly, we check that the defined correspondence

$$\{T\text{-fixed points in } \mathfrak{M}(n,k)\} \leftrightarrow \{n\text{-colored partitions } \boldsymbol{\lambda} \text{ of } k\} \tag{30}$$

really is one-to-one. We start with injectivity: Thus, suppose $(B_{1,2}, I, J)$ and $(B'_{1,2}, I', J')$ represent two fixed points corresponding to the same n-colored partition $\boldsymbol{\lambda}$. We want to show that they actually represent the same fixed point, which is the case if and only if they lie in the same $GL(k, \mathbb{C})$-orbit. We define a linear map $g \in GL(k, \mathbb{C})$ by specifying it on the bases $\mathcal{B}, \mathcal{B}'$ of K defined by $(B_{1,2}, I, J)$ and $(B'_{1,2}, I', J')$:

$$g : b_{\alpha\square} \mapsto b'_{\alpha\square},$$

i.e. g is the ordered basis transformation $\mathcal{B} \to \mathcal{B}'$. This definition is possible because $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$. For any basis element $b'_{\alpha\square} \in \mathcal{B}'$,

$$gB_1g^{-1}b'_{\alpha\square} = gB_1b_{\alpha\square} = gb_{\alpha,\square+(1,0)} = b'_{\alpha,\square+(1,0)} = B'_1b'_{\alpha\square}$$

(where $b_{\alpha,\square+(1,0)}$ is zero when $\square + (1,0)$ lies outside the Young diagram). This proves that $gB_1g^{-1} = B'_1$, and analogously $gB_2g^{-1} = B'_2$ holds too. Furthermore, using $I_\alpha = b_{\alpha,(1,1)}$ and $I'_\alpha = b'_{\alpha,(1,1)}$,

$$gI = g\begin{pmatrix} I_1 & \dots & I_n \end{pmatrix} = \begin{pmatrix} gI_1 & \dots & gI_n \end{pmatrix} = \begin{pmatrix} I'_1 & \dots & I'_n \end{pmatrix} = I'.$$

As, by Lemma 3.9, $J = J' = 0$, these are also g-related. Hence we have found that $(B_{1,2}, I, J)$ and $(B'_{1,2}, I', J')$ really lie in the same $GL(k, \mathbb{C})$-orbit, meaning that (30) is injective.

To see the surjectivity of (30), let $\boldsymbol{\lambda}$ be an n-colored partition of size $k = |\boldsymbol{\lambda}|$. Let

$$\mathcal{B} = \{b_{\alpha,\square} \mid \alpha = 1, \dots, n, \square \in \lambda^\alpha\}$$

be any basis of K parametrized by the boxes in $\boldsymbol{\lambda}$. If $\square \notin \lambda^\alpha$, we set $b_{\alpha\square} := 0$. Now we construct a fixed point $[B_{1,2}, I, J] \in \mathfrak{M}(n,k)$ corresponding to $\boldsymbol{\lambda}$ by specifying the linear maps $B_{1,2}, I, J$ on the basis $\mathcal{B}$:

$$\begin{aligned} B_1 : K \to K, &\quad b_{\alpha,\square} \mapsto b_{\alpha,\square+(1,0)} \\ B_2 : K \to K, &\quad b_{\alpha,\square} \mapsto b_{\alpha,\square+(0,1)} \\ I : N \to K, &\quad e_\alpha \mapsto b_{\alpha,(1,1)} \end{aligned}$$

where e_α, $\alpha = 1, \dots, n$ is the standard basis of $N = \mathbb{C}^n$. Also set $J = 0 : K \to N$. It is easy to see that $(B_{1,2}, I, J)$ satisfy the stability condition and $\mu^c = 0$, i.e. they represent a point in $\mathfrak{M}(n,k)$. To see that this point is T-fixed, define the map

$$\begin{gathered} \phi : \mathfrak{t} \to \mathfrak{gl}(k, \mathbb{C}), \quad (\mathfrak{a}, \vec{\varepsilon}) \mapsto \phi(\mathfrak{a}, \vec{\varepsilon}), \\ \phi(\mathfrak{a}, \vec{\varepsilon})b_{\alpha,(s_1,s_2)} := (\mathfrak{a}_\alpha + (s_1 - 1)\varepsilon_1 + (s_2 - 1)\varepsilon_2)b_{\alpha,(s_1,s_2)}. \end{gathered}$$

Then equations (25), (26), (27) hold, meaning that the infinitesimal actions of $(\mathfrak{a}, \vec{\varepsilon}) \in \mathfrak{t}$ and $\phi(\mathfrak{a}, \vec{\varepsilon}) \in \mathfrak{gl}(k, \mathbb{C})$ on $(B_{1,2}, I, J)$ are identical. This implies that the torus action on $(B_{1,2}, I, J)$ preserves the $GL(k, \mathbb{C})$-orbit. Thus, $[B_{1,2}, I, J] \in \mathfrak{M}(n, k)$ is the T-fixed point corresponding to $\boldsymbol{\lambda}$.

Now we have proved that the correspondence (30) is well-defined and 1-to-1 and thereby identified all T-fixed points of $\mathfrak{M}(n, k)$.

Theorem 3.11. *On the moduli space $\mathfrak{M}(n, k)$ defined in (18), the fixed points of the non-compact torus action given by (22), (23) are in 1-to-1 correspondence with the set $\mathfrak{P}(n, k)$ of n-colored partitions of size k.*

Furthermore, at the fixed point specified by $\boldsymbol{\lambda} \in \mathfrak{P}(n, k)$, the homomorphism $\phi : \mathfrak{t} \longrightarrow \mathfrak{gl}(k, \mathbb{C})$, given by the fixed point equation (24), has its eigenvalues parametrized by the boxes in the associated Young diagrams:

$$\phi_{\alpha,\square} = \mathfrak{a}_\alpha + (s_1 - 1)\varepsilon_1 + (s_2 - 1)\varepsilon_2$$

for $\alpha = 1, \ldots, n$ and $\square = (s_1, s_2) \in \lambda^\alpha$.

3.2.2 The tangent isotropy representations

From (12), our objective is the equivariant integral $\int_{\mathfrak{M}(n,k)} 1$ where $1 \in H_T^*(\mathfrak{M}(n, k); \mathbb{R})$ is the multiplicative identity of the equivariant cohomology in the Cartan model. For a T-fixed point $\boldsymbol{\lambda} = [B_{1,2}, I, J]$, the pullback $\iota_{\boldsymbol{\lambda}}^* 1 = 1 \in S(\mathfrak{t}^\vee)$. Thus, appealing to the equivariant localization Theorem 2.14, we define the integral

$$\int_{\mathfrak{M}(n,k)} 1 = \sum_{\boldsymbol{\lambda} \in \mathfrak{P}(n,k)} \frac{1}{\epsilon^T(T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k))}$$

as the sum over all n-colored partitions $\boldsymbol{\lambda}$ of size k, of the reciprocals of the Euler classes of the tangent isotropy T-representations. Our goal now is to compute this Euler class of $T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k)$ which, by Lemma 2.13, is the product of the representation's weights. Thus, the problem becomes describing the structure of the representation $T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k)$ in a way that reveals its weights.

As the manifold $\mathfrak{M}(n, k)$ is constructed as the quotient of a zero set $\{\mu^c = 0, \text{stability}\}$ by a group action $GL(k, \mathbb{C}) \curvearrowright X$, the tangent space $T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k)$ is canonically isomorphic to the cohomology of the complex of vector spaces

$$\mathfrak{gl}(k, \mathbb{C}) \xrightarrow{\alpha} T_{(B_{1,2},I,J)}X \xrightarrow{d\mu^c} \mathrm{End}(K), \tag{31}$$

where the first map is the infinitesimal action $\alpha : \mathfrak{gl}(k, \mathbb{C}) \longrightarrow T_{(B_{1,2},I,J)}X$ and the second map the derivative of the complex moment map: $d\mu^c_{(B_{1,2},I,J)} : T_{(B_{1,2},I,J)}X \longrightarrow \mathrm{End}(K)$. By the freeness of the $GL(k, \mathbb{C})$-action on X, the first map is injective. By Lemma 3.6, the second map is surjective. Our strategy in calculating $\epsilon^T(T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k))$ is then applying Lemma 2.16. But to apply this lemma, we need that the complex (31) is equivariant. We find T-representation structures on $\mathfrak{gl}(k, \mathbb{C})$, $T_{(B_{1,2},I,J)}X$ and $\mathrm{End}(K)$ such that the maps in (31) are equivariant and the T-representation $T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k)$, given by (22), (23), is the cohomology of the complex. Then Lemma 2.16 implies that

$$T_{\boldsymbol{\lambda}}\mathfrak{M}(n, k) \cong -\mathfrak{gl}(k, \mathbb{C}) + T_{(B_{1,2},I,J)}X - \mathrm{End}(K)$$

are isomorphic as virtual representations. We start by modifying the T-action on X (cf. (22), (23)) using the map $\phi : \mathfrak{t} \longrightarrow \mathfrak{gl}(k, \mathbb{C})$ characterizing the fixed point:

Lemma 3.12. *Suppose there are two commuting group actions G, T on a set X and the action of G is free. For any point $[x] \in X/G$ that is fixed under the descending action $T \curvearrowright X/G$, there is a unique antihomomorphism $g : T \to G$ such that $tx = gx$ for all $t \in T$. Furthermore, the modified T-action*

$$T \curvearrowright X : \quad t * x := g(t)^{-1} tx$$

on X and the unmodified T-action $(t, x) \mapsto tx$ on X descend to the same T-action on the quotient X/G.

Proof. The descending T-action on X/G is defined by $t[x] = [tx]$. Thus $[x] \in X/G$ is T-fixed if and only if for each $t \in T$ there is a $g \in G$ such that $tx = gx$. As the G-action is free, this defines a function $g : T \to G$. Now for any $t_1, t_2 \in T$ it holds that $t_1 t_2 x = t_1 g(t_2) x = g(t_2) t_1 x = g(t_2) g(t_1) x$, thus g is an antihomomorphism, which makes the modified T-action $T \curvearrowright X : t * x := g(t)^{-1} tx$ well-defined. While the modified T-action and the G-action on X don't necessarily commute, the modified T-action still descends to the quotient X/G:

$$t * gx = g(t)^{-1} tgx = g(t)^{-1} gtx \sim g(t)^{-1} tx = t * x,$$

where $\sim$ is the equivalence relation on X defined by the G-action. As $t * x = g(t)^{-1} tx \sim tx$, the modified and unmodified T-actions really do descend to the same action on X/G. □

If one instead defines $g(t)$ by $g(t)tx = x$, it would be a homomorphism. In our application T is abelian so the antihomomorphism is a homomorphism anyway. The modified T-action on X, defined by Lemma 3.12 and homomorphism $\phi : \mathfrak{t} \to \mathfrak{gl}(k, \mathbb{C})$:

$$\begin{aligned} e^{(\mathfrak{a}, \vec{\varepsilon})} * (B_{1,2}, I, J) &= e^{-\phi(\mathfrak{a}, \vec{\varepsilon})} \cdot \left(e^{(\mathfrak{a}, \vec{\varepsilon})} \cdot (B_{1,2}, I, J) \right) \\ &= (e^{\varepsilon_1} e^{-\phi} B_1 e^{\phi}, e^{\varepsilon_2} e^{-\phi} B_2 e^{\phi}, e^{-\phi} I e^{\mathfrak{a}}, e^{\varepsilon} e^{-\mathfrak{a}} J e^{\phi}), \end{aligned} \tag{32}$$

where $\phi(\mathfrak{a}, \vec{\varepsilon})$ satisfies the fixed point equations (25), (26), (27) and has spectrum given by $\boldsymbol{\lambda}$ as in Theorem 3.11. This action is in fact linear, so the isotropy representation on $T_{(B_{1,2}, I, J)} X$ looks identical.

From Lemma 3.12 it is clear that the action induced on the quotient $\mathfrak{M}(n, k)$ by the modified action on X is identical to that induced by the unmodified action. The same is true for the isotropy representations on $T_{\boldsymbol{\lambda}} \mathfrak{M}(n, k)$, whose structure is our objective. From now on, we always regard $T_{(B_{1,2}, I, J)} X$ as the modified T-representation. We define auxiliary T-representations

- on K by $e^{(\mathfrak{a}, \vec{\varepsilon})} * v = e^{\phi} v$ (this depends on the fixed point $\boldsymbol{\lambda}$, through ϕ, and is isomorphic to the representation $K_{\boldsymbol{\lambda}}$ defined in the appendix A using the quotient $\mathbb{C}[x, y]/I_{\lambda}$),
- on N by $e^{(\mathfrak{a}, \vec{\varepsilon})} * w = e^{\mathfrak{a}} w$,
- on $Q_1 = \mathbb{C}$ by $e^{(\mathfrak{a}, \vec{\varepsilon})} * z = e^{\varepsilon_1} z$ and on $Q_2 = \mathbb{C}$ by $e^{(\mathfrak{a}, \vec{\varepsilon})} * z = e^{\varepsilon_2} z$. Their sum is $Q := Q_1 + Q_2$.

Then, looking at (32), there is an isomorphism of representations

$$T_{(B_{1,2}, I, J)} X \cong Q_1 K^* K + Q_2 K^* K + K^* N + Q_1 Q_2 N^* K$$

where we drop from our notation tensor products and write + for direct sums.

Lemma 3.13. *There exist T-representation structures on $\mathfrak{gl}(k,\mathbb{C})$ and $\mathrm{End}(K)$ such that*

$$\mathfrak{gl}(k,\mathbb{C}) \xrightarrow{\alpha} T_{(B_{1,2},I,J)}X \xrightarrow{d\mu^c} \mathrm{End}(K)$$

is a complex of representations, meaning in particular that α and $d\mu^c$ are T-equivariant. These representations are isomorphic to:

$$\mathfrak{gl}(k,\mathbb{C}) \cong K^*K, \qquad \mathrm{End}(K) \cong Q_1Q_2K^*K$$

We deliberately use two different notations $\mathfrak{gl}(k,\mathbb{C})$, $\mathrm{End}(K)$ for the same vector space to stress that they are different as representations.

Proof. The map $\mathfrak{gl}(k,\mathbb{C}) \to T_{(B_{1,2},I,J)}X$ is the infinitesimal $GL(k,\mathbb{C})$-action on X, i.e.

$$\psi \mapsto ([\psi, B_{1,2}], \psi I, -J\psi).$$

The representation structure on $T_{(B_{1,2},I,J)}X$ is given by (32). Using the fixed point equation $e^{(\mathfrak{a},\vec{\varepsilon})} * (B_{1,2},I,J) = (B_{1,2},I,J)$, it is then easy to check that the map is equivariant if we equip $\mathfrak{gl}(k,\mathbb{C})$ with the T-representation structure $e^{(\mathfrak{a},\vec{\varepsilon})} * M = e^{-\phi}Me^{\phi}$.

We want the map

$$d\mu^c_{(B_{1,2},I,J)} : T_{(B_{1,2},I,J)}X \to \mathrm{End}(K), \quad (\delta B_{1,2}, \delta I, \delta J) \mapsto [\delta B_1, B_2] + [B_1, \delta B_2] + (\delta I)J + I(\delta J)$$

to be equivariant. Let us see how $d\mu^c$ transforms under the modified T-action on $T_{(B_{1,2},I,J)}X$:

$$d\mu^c_{(B_{1,2},I,J)}(e^{\varepsilon_{1,2}}e^{-\phi}\delta B_{1,2}e^{\phi}, e^{-\phi}\delta I e^{\mathfrak{a}}, e^{\varepsilon}e^{-\mathfrak{a}}\delta J e^{\phi}) = e^{\varepsilon}e^{-\phi}d\mu^c_{(B_{1,2},I,J)}(\delta B_{1,2}, \delta I, \delta J)e^{\phi},$$

where we used the fixed point equation $e^{(\mathfrak{a},\vec{\varepsilon})} * (B_{1,2},I,J) = (B_{1,2},I,J)$. This means that, if we define the representation structure of $\mathrm{End}(K)$ to be $Q_1Q_2K^*K$, then $d\mu^c_{(B_{1,2},I,J)}$ is T-equivariant. □

Thus, applying Lemma 2.16 shows that the T-representation $T_\lambda\mathfrak{M}(n,k)$ is isomorphic to the virtual representation

$$T_\lambda\mathfrak{M}(n,k) \cong -K^*K + Q_1K^*K + Q_2K^*K + K^*N + Q_1Q_2N^*K - Q_1Q_2K^*K \tag{33}$$

Defining $Q = Q_1 + Q_2$ and $\wedge_{-1}Q = \wedge^0 Q - \wedge^1 Q + \wedge^2 Q$, this can be written also as

$$K^*N + Q_1Q_2N^*K - (\wedge_{-1}Q)K^*K. \tag{34}$$

Remark 3.14. The representations Q, K, N come from T-equivariant vector bundles over $\mathfrak{M}(n,k)$:

- $\mathbf{Q}$ is the spacetime bundle, a trivial complex vector bundle of rank two defined as the pullback by the collapsing map $\mathfrak{M}(n,k) \to *$ of the representation Q. It splits $\mathbf{Q} = \mathbf{Q}_1 \oplus \mathbf{Q}_2$.

- $\mathbf{N}$ is the framing bundle, a trivial complex vector bundle of rank n defined as the pullback by the collapsing map of the representation N.

- We define the instanton bundle $\mathbf{K}$ to be the associated vector bundle to the principal $GL(k,\mathbb{C})$-bundle
$$\{\mu^c = 0,\ \text{stability}\} \to \{\mu^c = 0,\ \text{stability}\}/GL(k,\mathbb{C})$$
and the fundamental representation of $GL(k,\mathbb{C})$.

The representations Q, N, K are then the isotropy representations of these bundles at the fixed point $\boldsymbol{\lambda}$, i.e. $Q = \mathbf{Q}|_{\boldsymbol{\lambda}}$, $N = \mathbf{N}|_{\boldsymbol{\lambda}}$ and $K = \mathbf{K}|_{\boldsymbol{\lambda}}$.

For the purposes of quiver gauge theories, which have matter fields, we will later also introduce equivariant vector bundles $\mathbf{M}$ encoding the masses of the theory and these will contribute to the integrand of the integral over the moduli space.

It would now be nice to simply compute the Euler class of $T_{\boldsymbol{\lambda}}\mathfrak{M}(n,k)$ using (34) as
$$\frac{\epsilon^T(K^*N)\,\epsilon^T(Q_1Q_2N^*K)}{\epsilon^T(\wedge_{-1}K^*K)}.$$
However, some of the representations in (34) have zero as one of their weights, meaning that their Euler class vanishes. The zero weight spaces (as well as many other weight spaces) in (34) cancel in the difference, simplifying as stated in the following lemma.

Lemma 3.15 ([NY05]). *The T-representation $T_{\boldsymbol{\lambda}}\mathfrak{M}(n,k)$ is isomorphic to*
$$\sum_{\alpha,\beta=1}^{n} e^{\mathfrak{a}_\alpha-\mathfrak{a}_\beta}\left(\sum_{\square\in\lambda^\alpha} Q_1^{-l_{\lambda^\beta}(\square)}Q_2^{a_{\lambda^\alpha}(\square)+1} + \sum_{\square\in\lambda^\beta} Q_1^{l_{\lambda^\alpha}(\square)+1}Q_2^{-a_{\lambda^\beta}(\square)}\right), \tag{35}$$
where $e^{\mathfrak{a}_\alpha-\mathfrak{a}_\beta}$ denotes a one-dimensional T-representation with character $e^{\mathfrak{a}_\alpha-\mathfrak{a}_\beta}$, the powers of $Q_{1,2}$ are to be understood as tensor products, and the arm- and leg-length of a box $\square = (s_1, s_2)$ in a partition λ is defined as $a_\lambda(\square) = \lambda_{s_1} - s_2$ and $l_\lambda(\square) = \check{\lambda}_{s_2} - s_1$.

Proof. We already showed that $T_{\boldsymbol{\lambda}}\mathfrak{M}(n,k)$ is isomorphic to the virtual representation (33). The representations involved in this sum split into their weight-spaces according to
$$K = \sum_{\alpha=1}^{n} e^{\mathfrak{a}_\alpha}\sum_{\square\in\lambda^\alpha} Q_1^{s_1-1}Q_2^{s_2-1}, \qquad N = \sum_{\alpha=1}^{n} e^{\mathfrak{a}_\alpha}.$$
Proving the lemma then boils down to cancelling the weight spaces that occur in (33) with both a positive and a negative sign. This process is well-presented in [NY05, Thm. 2.11], so we skip it here. □

Now we just need to compute the Euler classes, i.e. the products of the weights, of (35). Using that the weight of the one-dimensional representation $e^{\mathfrak{a}_\alpha-\mathfrak{a}_\beta}Q_1^iQ_2^j$ is $\mathfrak{a}_\alpha - \mathfrak{a}_\beta + i\varepsilon_1 + j\varepsilon_2$, we can finally evaluate the partition function $\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty}\mathfrak{q}^k\sum_{\boldsymbol{\lambda}\in\mathfrak{P}(n,k)} Z_{\boldsymbol{\lambda}}$ completely:
$$\begin{aligned} Z_{\boldsymbol{\lambda}} &= \frac{1}{\epsilon^T(T_{\boldsymbol{\lambda}}\mathfrak{M}(n,k))} \\ &= \prod_{\alpha,\beta=1}^{n}\prod_{\square\in\lambda^\alpha}\frac{1}{\mathfrak{a}_\alpha - \mathfrak{a}_\beta - l_{\lambda^\beta}(\square)\varepsilon_1 + (a_{\lambda^\alpha}(\square)+1)\varepsilon_2}\prod_{\square\in\lambda^\beta}\frac{1}{\mathfrak{a}_\alpha - \mathfrak{a}_\beta + (l_{\lambda^\alpha}(\square)+1)\varepsilon_1 - a_{\lambda^\beta}(\square)\varepsilon_2} \end{aligned} \tag{36}$$

In [Nek03] and [NO06], the expression (36) was simplified:

$$\begin{aligned} Z_{\boldsymbol{\lambda}} &= \prod_{\alpha,\beta;s,t} \frac{\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \varepsilon_1(s-1) + \varepsilon_2(-t)}{\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \varepsilon_1(\check{\lambda}_t^\alpha - s) + \varepsilon_2(t - \lambda_s^\beta - 1)} \\ &= \prod_{(\alpha,s)\neq(\beta,t)} \frac{\Gamma(\lambda_s^\alpha - \lambda_t^\beta + \nu(t-s+1) + \mathfrak{b}_{\alpha\beta})\Gamma(\nu(t-s) + \mathfrak{b}_{\alpha\beta})}{\Gamma(\lambda_s^\alpha - \lambda_t^\beta + \nu(t-s) + \mathfrak{b}_{\alpha\beta})\Gamma(\nu(t-s+1) + \mathfrak{b}_{\alpha\beta})}, \end{aligned} \tag{37}$$

where the products are over $\alpha, \beta \in \{1, \ldots, n\}$ and $s, t \in \mathbb{Z}_{>0}$ (not just boxes within the partition but also those outside of it) and we define $\nu = -\varepsilon_1/\varepsilon_2$ and $\mathfrak{b}_{\alpha\beta} = \frac{\mathfrak{a}_\alpha - \mathfrak{a}_\beta}{2}$. We should point out that these simplifications only hold for $n \geq 2$.

For the special case where $\nu = 1$, we adopt the notation $\hbar \coloneqq -\varepsilon_1 = \varepsilon_2$. In this case, (37) yields

$$Z_{\boldsymbol{\lambda}} = \prod_{(\alpha,s)\neq(\beta,t)} \frac{\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \hbar(\lambda_s^\alpha - \lambda_t^\beta + t - s)}{\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \hbar(t-s)}.$$

In this case, $Z_{\boldsymbol{\lambda}}$ is related to the Plancherel measure on the set of isomorphism types of irreducible representations of the symmetric group: The irreducible representations of the symmetric group S_k are classified by the partitions λ of k. More details are given in [NO06].

3.2.3 Other results

Approach of Nakajima–Yoshioka. Below (18) we discussed that $\mathfrak{M}(n,k)$ can be interpreted as the moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank n and second Chern number k. In [NY05], Nakajima and Yoshioka, working within this algebro-geometric framework, identified the fixed points of the T-action on $\mathfrak{M}(n,k)$ and computed the T-representation structure of the tangent spaces and from this the Nekrasov partition function. In this setting, fixed points are sheaves $I_1 \oplus \cdots \oplus I_n$ where each $I_\alpha \in \mathfrak{M}(1, k_\alpha)$ is an ideal sheaf of a zero-dimensional subscheme contained in $\mathbb{P}^2 \setminus \ell_\infty = \mathbb{C}^2$, and a fixed point in $\mathfrak{M}(1, k_\alpha)$. In [ES87] it was shown that $I_\alpha \in \mathfrak{M}(n, k_\alpha)$ is fixed if and only if (identifying it with its ring of sections on $\mathbb{C}^2$), as an ideal of $\mathbb{C}[x,y]$, it is generated by monomials $x^a y^b$. Through the construction in the appendix A, it then corresponds to a partition.

In their paper, Nakajima and Yoshioka also proved Nekrasov's [Nek03] conjecture that $F_{\text{inst}}(\mathfrak{a}, \vec{\varepsilon}) \coloneqq \varepsilon_1\varepsilon_2 \log \mathcal{Z}_{\text{inst}}(\mathfrak{a}, \vec{\varepsilon})$ is regular at $\vec{\varepsilon} = 0$ and that $F_{\text{inst}}(\mathfrak{a}, 0, 0)$ is the instanton part of the Seiberg–Witten prepotential. They did this by performing equivariant localization for a second moduli space $\hat{\mathfrak{M}}$, the framed moduli space of torsion free sheaves on $\hat{\mathbb{P}}^2$, the blowup of $\mathbb{P}^2$ at $[1:0:0]$. This formally computes the integral

$$\hat{Z} = \int_{\hat{\mathfrak{M}}} 1,$$

defining the partition function on the blowup. To prove the regularity of F_{inst}, they exploited relationships between the partition functions $Z(\mathfrak{a}, \vec{\varepsilon})$ and $\hat{Z}(\mathfrak{a}, \vec{\varepsilon})$.

Contour integral formula. Each $Z_k = \sum_{k[\boldsymbol{\lambda}]=k} Z_{\boldsymbol{\lambda}}$ can be expressed as the multi-dimensional contour integral

$$Z_k = \frac{1}{k!} \frac{\varepsilon^k}{(\varepsilon_1\varepsilon_2)^k} \frac{1}{(2\pi i)^k} \oint \prod_{s=1}^k \frac{d\phi_s}{\mathcal{A}(\phi_s)\mathcal{A}(\phi_s + \varepsilon)} \prod_{1 \leq s < t \leq k} \frac{\phi_{s,t}^2(\phi_{s,t}^2 - \varepsilon^2)}{(\phi_{s,t}^2 - \varepsilon_1^2)(\phi_{s,t}^2 - \varepsilon_2^2)}$$

where $\mathcal{A}(x) = \prod_{s=1}^{n}(x - \mathfrak{a}_s)$ and $\phi_{s,t} = \phi_s - \phi_t$. Nekrasov first obtained this formula in his original paper on instanton counting [Nek03], using a different localization technique developed in a different physical context [MNS00]. Within his paper, he doesn't conduct the fixed point analysis as we did but rather appeals to the contour integral, whose evaluation using the residue theorem yields a sum over poles, the poles corresponding to fixed points or partitions.

3.2.4 Instanton counting in equivariant K-theory

The K-theory partition function is

$$\mathcal{Z}_{\text{inst}}^{K} = \pi_* \epsilon^T(0)$$

where 0 is the rank-zero equivariant vector bundle over $\mathfrak{M}(n,k)$, ϵ^T the K-theory equivariant Euler class, i.e. $\epsilon^T(0) = 1$ is the multiplicative identity of $K_T(\mathfrak{M}(n,k))$ and $\pi_* : K_T(\mathfrak{M}(n,k)) \to K_T(*) \cong \mathbb{C}[T]$ is induced by the collapsing map. Formally applying the localization formula 2.19, we obtain $\mathcal{Z}_{\text{inst}}^{K} = \sum_{k=0}^{\infty} \mathfrak{q}^k \sum_{\boldsymbol{\lambda} \in \mathfrak{P}(n,k)} Z_{\boldsymbol{\lambda}}^{K}$ where

$$\begin{aligned} Z_{\boldsymbol{\lambda}}^{K} &= \frac{1}{\epsilon^T(T_{\boldsymbol{\lambda}}\mathfrak{M}(n,k))} \\ &= \prod_{\alpha,\beta=1}^{n} \prod_{\square \in \lambda^\alpha} \frac{1}{1 - e^{-\mathfrak{a}_\alpha + \mathfrak{a}_\beta + l_{\lambda^\beta}(\square)\varepsilon_1 - (a_{\lambda^\alpha}(\square)+1)\varepsilon_2}} \prod_{\square \in \lambda^\beta} \frac{1}{1 - e^{-\mathfrak{a}_\alpha + \mathfrak{a}_\beta - (l_{\lambda^\alpha}(\square)+1)\varepsilon_1 + a_{\lambda^\beta}(\square)\varepsilon_2}} \end{aligned} \tag{38}$$

Note the similarity between (36) and (38): To get from the equivariant cohomology formula to the K-theory formula, one need only replace all factors X in $Z_{\boldsymbol{\lambda}}$ by factors $1 - e^{-X}$ in $Z_{\boldsymbol{\lambda}}^{K}$. This can again be simplified further (case $-\varepsilon_1 = \varepsilon_2 = \hbar$) [Nek03]:

$$Z_{\boldsymbol{\lambda}}^{K} = \prod_{(\alpha,s)\neq(\beta,t)} \frac{\sinh \frac{1}{2}(\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \hbar(\lambda_s^\alpha - \lambda_t^\beta + t - s))}{\sinh \frac{1}{2}(\mathfrak{a}_\alpha - \mathfrak{a}_\beta + \hbar(t-s))}$$

Physically, the K-theory partition function corresponds to five-dimensional gauge theories compactified on a circle. If one introduces a parameter β proportional to the circle's radius of, then $\frac{1}{2}$ gets replaced by $\frac{\beta}{2}$ in the equation above. Then $\lim_{\beta\to 0} Z_{\boldsymbol{\lambda}}^{K} = Z_{\boldsymbol{\lambda}}$, meaning that the cohomology partition function is the limit of the K-theory partition function as the length of the fifth dimension goes to zero [Nek03, Sec. 4].

3.3 $\mathcal{N} = 2$ quiver gauge theories

We want to generalize instanton counting to gauge theories on Euclidean spacetime $\mathbb{R}^4$ characterized by quivers, whose vertices and edges correspond to fields.

Definition 3.16. *A quiver γ is a directed graph with set of vertices* Vert_γ *and set of directed edges* Edges_γ.

For our purposes all quivers will be connected and have only finitely many vertices and edges. We write

$$e : i \to j,$$

where $e \in \text{Edges}_\gamma$ and $i, j \in \text{Vert}_\gamma$, to indicate that e is an edge directed from vertex i (its source) to vertex j (its target). For source i and target j we also use the notation $i = \text{in}(e)$, $j = \text{out}(e)$.

Quivers will be used to define gauge theories including massive fields. For a given quiver, to define the associated theory's gauge group, we additionally need a coloring $\underline{n} = (n_i)_{i\in \mathrm{Vert}_\gamma} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$ of the quiver. The gauge group is then the product

$$G = \bigtimes_{i\in \mathrm{Vert}_\gamma} U(n_i),$$

so each vertex contributes a factor to the total gauge group. For each factor $U(n_i)$ of the gauge group, we have an exponentiated coupling constant $\mathfrak{q}_i \in \mathbb{C}^\times$. The collection of coupling constants is denoted by $\underline{\mathfrak{q}} = (\mathfrak{q}_i)_{i\in \mathrm{Vert}_\gamma}$.

To define the theory's matter field content, we introduce another coloring $\underline{m} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$. The theory's flavor symmetry group is

$$G_f = \left(\Big(\bigtimes_{i\in \mathrm{Vert}_\gamma} U(m_i)\Big) \times U(1)^{\mathrm{Edges}_\gamma}\right) \Big/ U(1)^{\mathrm{Vert}_\gamma} \tag{39}$$

where we quotient by the $U(1)^{\mathrm{Vert}_\gamma}$-action (defining a normal subgroup)

$$(u_i)_{i\in\mathrm{Vert}_\gamma} : \big((g_i)_{i\in\mathrm{Vert}_\gamma}, (u_e)_{e\in\mathrm{Edges}_\gamma}\big) \mapsto \big((u_i g_i)_{i\in\mathrm{Vert}_\gamma}, (u_{\mathrm{in}(e)} u_e u_{\mathrm{out}(e)}^{-1})_{e\in\mathrm{Edges}_\gamma}\big).$$

The theory's field content is determined by the quiver and its colorings:

- for each vertex $i \in \mathrm{Vert}_\gamma$ a set of $\mathcal{N} = 2$ vector multiplets transforming in the adjoint representation of (its factor $U(n_i)$ of) the gauge group

- for each vertex $i \in \mathrm{Vert}_\gamma$ a set of $\mathcal{N} = 2$ hypermultiplets transforming in the fundamental representation $\mathbb{C}^{n_i}$ of the gauge group and the antifundamental representation $\overline{\mathbb{C}}^{m_i}$ of the flavor symmetry group

- for each edge $e \in \mathrm{Edges}_\gamma$ a set of $\mathcal{N} = 2$ hypermultiplets transforming in the bifundamental representation $\overline{\mathbb{C}}^{n_{\mathrm{in}(e)}} \otimes \mathbb{C}^{n_{\mathrm{out}(e)}}$ of the gauge group. In the case where the edge is an edge loop, i.e. $e : i \longrightarrow i$ connects a vertex to itself, this is a hypermultiplet in the adjoint representation.

For a definition of vector multiplets and hypermultiplets, we refer to [Bil01, Chap. 3]. We will not work directly with these fields. Rather, we will define the theory's instanton partition function geometrically. The quiver gauge theory's resolved instanton moduli space $\mathfrak{M}(\underline{n}, \underline{k})$ will be defined in (41). The instanton partition function is then the generating function of equivariant integrals of certain vector bundles' equivariant Euler classes over this moduli space, as we will discuss in Section 3.4. As the theory is fully determined by the quiver and the two integer colorings $\underline{n} > 0$, $\underline{m} \geq 0$, we will often simply refer to triples $(\gamma, \underline{n}, \underline{m})$ as quiver gauge theories.

The theory also has a rotational symmetry $G_{\mathrm{rot}} = \mathrm{Spin}(4) \cong SU(2)_L \times SU(2)_R$. Its maximal torus is $U(1)^2$. The complexification of the full symmetry group $H = G \times G_f \times G_{\mathrm{rot}}$ acts on the moduli space $\mathfrak{M}(\underline{n}, \underline{k})$. Restricting the action to the complexification $T = T_G \times T_{G_f} \times T_{G_{\mathrm{rot}}}$ of the maximal torus of H, we will perform equivariant localization. This reduces the partition function to a sum of rational functions of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$. Elements of $\mathfrak{t}$, the equivariant parameters, will be denoted

$$(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon}) \in \mathfrak{t}$$

and are physically interpreted as the Coulomb moduli (vacuum expectations of scalar Higgs fields), masses and parameters of the Ω-background (cf. [Ell15]). The Coulomb moduli

$$\underline{\mathfrak{a}} = (\mathfrak{a}_i)_{i\in\mathrm{Vert}_\gamma} \in \mathrm{Lie}(T_G) = \bigoplus_{i\in\mathrm{Vert}_\gamma} \mathfrak{gl}(1,\mathbb{C})^{n_i}, \qquad \mathfrak{a}_i = \begin{pmatrix} \mathfrak{a}_{i,1} & & \\ & \ddots & \\ & & \mathfrak{a}_{i,n_i} \end{pmatrix}$$

consist of diagonal matrices with entries $\mathfrak{a}_{i,\alpha}$. The masses are split

$$\underline{\mathfrak{m}} = (\mathfrak{m}_i)_{i\in\mathrm{Vert}_\gamma} \oplus (\mathfrak{m}_e)_{e\in\mathrm{Edges}_\gamma} \in \bigoplus_{i\in\mathrm{Vert}_\gamma} \mathfrak{gl}(1,\mathbb{C})^{m_i} \oplus \bigoplus_{e\in\mathrm{Edges}_\gamma} \mathfrak{gl}(1,\mathbb{C}).$$

Note that the RHS is not quite the Lie algebra of T_{G_f}, as G_f (cf. (39)) is a quotient by $U(1)^{\mathrm{Vert}_\gamma}$, so there is a constraint on $\underline{\mathfrak{m}}$. In the important case of the A_1 quiver (one vertex, no edges), this is $\sum_{f=1}^m \mathfrak{m}_f = 0$ (for one-vertex quivers we omit the vertex index). For each vertex the masses form a diagonal matrix, for edges they are just one scalar:

$$\mathfrak{m}_i = \begin{pmatrix} \mathfrak{m}_{i,1} & & \\ & \ddots & \\ & & \mathfrak{m}_{i,m_i} \end{pmatrix} \in \mathfrak{gl}(1,\mathbb{C})^{m_i}, \qquad \mathfrak{m}_e \in \mathfrak{gl}(1,\mathbb{C})$$

The parameters of the Ω-background are

$$\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathrm{Lie}(T_{G_{\mathrm{rot}}}) = \mathfrak{gl}(1,\mathbb{C})^2.$$

We frequently use the notation $\varepsilon = \varepsilon_1 + \varepsilon_2$.

Examples. We list the two basic examples of quiver gauge theories. The A_N quiver ($N \geq 1$) has set of vertices $\mathrm{Vert}_\gamma = \{1, \dots, N\}$ and edges $\mathrm{Edges}_\gamma = \{1, \dots, N-1\}$ where the source of edge e is vertex e and its target is vertex $e+1$.

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow N$$

In particular, the A_1 quiver is the quiver with just one vertex and no edges. The quiver gauge theory $(A_1, n, 0)$ is exactly the $\mathcal{N} = 2$ pure (i.e. massless) $U(n)$ gauge theory for which we performed instanton counting in Section 3.2.

The $\hat{A}_N$ quiver ($N \geq 0$) is

$$\begin{tikzcd} & 0 \arrow[dl] & & \\ 1 \arrow[r] & 2 \arrow[r] & \dots \arrow[r] & N \arrow[ull, bend right] \end{tikzcd}$$

In particular, the $\hat{A}_0$ quiver is the quiver with just one vertex and one edge loop connecting the vertex to itself. It is also called the Jordan quiver.

Dynkin diagrams and quivers. More generally, one may consider quivers associated to simply-laced Dynkin diagrams (ADE) or their affinizations ($\hat{A}\hat{D}\hat{E}$). One is then confronted with choosing directions for the edges. The partition function depends on these directions.

We define a quiver's Cartan matrix:

$$c_{ij} := 2\delta_{ij} - \#\{e : i \to j\} - \#\{e : j \to i\}, \quad i, j \in \mathrm{Vert}_\gamma$$

This is the same as the matrix $2I - A$, where I is the identity matrix and A the adjacency matrix of the underlying undirected graph. It is always symmetric and doesn't depend on the directions of the edges in γ. When the underlying graph of γ is a simply-laced Dynkin diagram (ADE) or its affinization ($\hat{A}\hat{D}\hat{E}$), the quiver's Cartan matrix is the same as the Dynkin diagram's or the simple or affine Lie algebra's Cartan matrix.

Unlike for simply-laced Dynkin diagrams, we do not associate quivers to non-simply-laced Dynkin diagrams. Take for example $B_2 = \bullet\!\!\Rightarrow\!\!\bullet$. It is tempting to associate to it the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd}$$

However, the Cartan matrix of B_2 is $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$ while that of the quiver is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. More generally, all non-simply-laced Dynkin diagrams have asymmetric Cartan matrix, while all quivers have symmetric Cartan matrix. If we associate a quiver to a Dynkin diagram, their Cartan matrices should match. Thus, quivers can't correspond to non-simply-laced Dynkin diagrams.

Choices of gauge group. We defined the gauge group as a product of factors $U(n_i)$. One can just as well work with $SU(n_i)$ instead, and the instanton partition function would have the same form. The only difference is that the Coulomb moduli $\mathfrak{a}_i \in \mathfrak{gl}(n_i, \mathbb{C})$ would instead be restricted to the smaller Lie algebra $\mathfrak{sl}(n_i, \mathbb{C})$ (the complexification of $\mathfrak{su}(n_i, \mathbb{C})$). This is to say that the partition function of SU quiver gauge theory is the restriction of that of U quiver gauge theory.

Instanton counting has also been generalized to some other gauge groups [MW04; NS04].

Asymptotic freedom and conformality. When the gauge group is $\bigtimes_i SU(n_i)$, it can be shown [NP+23; Nov+83] that the beta function $\underline{\beta} = (\beta_i)_{i \in \mathrm{Vert}_\gamma}$ of the quiver gauge theory $(\gamma, \underline{n}, \underline{m})$ is

$$\beta_i = -2n_i + m_i + \sum_{e: i \to j} n_j + \sum_{e: j \to i} n_j, \tag{40}$$

which can also be written in terms of the quiver's Cartan matrix:

$$\beta_i = m_i - \sum_j c_{ij} n_j$$

The beta function encodes how the coupling constant of a theory changes with the action scale through renormalization. The theory is asymptotically conformal if $\underline{\beta} = 0$, and asymptotically free if $\underline{\beta} < 0$, by which we mean $\beta_i < 0$ for all $i \in \mathrm{Vert}_\gamma$. Imposing the condition $\underline{\beta} \leq 0$ constrains the quiver:

Proposition 3.17. *If a quiver gauge theory $(\gamma, \underline{n}, \underline{m})$ has $\underline{\beta} \leq 0$, then γ is of type ADE or $\hat{A}\hat{D}\hat{E}$.*

Colorings $\underline{n} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$, $\underline{m} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ such that $(\gamma, \underline{n}, \underline{m})$ has $\underline{\beta} \leq 0$ are guaranteed to exist for all ADE or $\hat{A}\hat{D}\hat{E}$ quivers:

- *If γ is an ADE quiver, then for every matter coloring $\underline{m} \geq 0$ exists a coloring $\underline{n} > 0$ such that the quiver gauge theory $(\gamma, \underline{n}, \underline{m})$ has β function $\underline{\beta} \leq 0$.*

- *If γ is a $\hat{A}\hat{D}\hat{E}$ quiver, then any quiver gauge theory $(\gamma, \underline{n}, \underline{m})$ with $\underline{\beta} \leq 0$ has that $\underline{\beta} = 0$ and the matter coloring $\underline{m}$ vanishes. Colorings $\underline{n} > 0$ such that $(\gamma, \underline{n}, 0)$ has $\underline{\beta} \leq 0$ exist and they are all related by multiplication with a positive integer.*

Proof. For the first claim, note that a quiver gauge theory with $\underline{\beta} \leq 0$ satisfies $\sum_j c_{ij} n_j \geq m_i$. As the matter coloring m_i is always non-negative, this implies $\sum_j c_{ij} n_j \geq 0$. Now, a basic fact from the theory of generalized Cartan matrices (cf. [Kac90, Chap. 4]) is that, if there exists a coloring $\underline{n} > 0$ such that $\sum_j c_{ij} n_j \geq 0$, then the Cartan matrix is positive semi-definite. This in turn implies that the underlying undirected graph is a simply-laced Dynkin diagram (positive definite) or its affinization (positive semi-definite) [BH11, Theorem 3.1.3].

For a finite Dynkin quiver (ADE), it can be seen from [Kac90, Thm. 4.3] that for any $\underline{m} \geq 0$ a coloring $\underline{n} > 0$ exists such that $\sum_j c_{ij} n_j \geq m_i$, i.e. $\underline{\beta} \leq 0$.

For an affine Dynkin quiver ($\hat{A}\hat{D}\hat{E}$), every coloring $\underline{n} > 0$ with $\sum_j c_{ij} n_j \geq 0$ in fact has $\sum_j c_{ij} n_j = 0$ [Kac90, Thm. 4.3]. Thus, any theory with $\underline{\beta} \leq 0$ must in fact have $\underline{\beta} = 0$ and $\underline{m} = 0$. As an affine Cartan matrix has one-dimensional kernel spanned by a vector with only positive real entries [Kac90, Thm. 4.3], and the Cartan matrix has integer entries, there exists a vector $\underline{n} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$ spanning the kernel of (c_{ij}). □

Thus there are strong physical arguments for restricting to quivers coming from finite or affine simply-laced Dynkin diagrams, which we will do in parts of Chapter 4. However, for now we perform instanton counting for general quivers.

3.4 Instanton counting for quiver gauge theories

In Section 3.2, we performed the equivariant localization for the case of gauge theories with single-factor gauge group $U(n)$ and with no matter fields, that is, the A_1 quiver gauge theories with matter coloring $m = 0$. The quiver gauge theories that give rise to non-perturbative Dyson–Schwinger equations and qq-characters are in general more complex. Here there are multi-factor gauge groups $\bigtimes_{i \in \mathrm{Vert}_\gamma} U(n_i)$ and matter fields; for each vertex there are m_i masses of an adjoint hypermultiplet, and for each edge $e : i \to j$ there is a hypermultiplet of mass $\mathfrak{m}_e$ in the bifundamental representation of the gauge group factors $U(n_i), U(n_j)$.

An instanton for the multi-factor gauge group consists of an instanton for each factor $U(n_i)$, each having its own charge. Thus, we define the resolved moduli space of instantons with charges $\underline{k} = (k_i)_{i \in \mathrm{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ as

$$\mathfrak{M}(\underline{n}, \underline{k}) \coloneqq \bigtimes_{i \in \mathrm{Vert}_\gamma} \mathfrak{M}(n_i, k_i). \tag{41}$$

It should be noted that this is not the quiver variety (which will be introduced in Section 4.4). The quiver's edges do not contribute to $\mathfrak{M}(\underline{n}, \underline{k})$. They will, however, contribute to the integrand of the integral defining the partition function (43).

Due to the multi-factor gauge group and the flavor symmetry group of the matter fields, the full group whose action on $\mathfrak{M}(\underline{n}, \underline{k})$ we consider is larger than in the previous section. Namely, it is $G \times G_f \times G_{\mathrm{rot}}$ which has as its maximal torus $U(1)^{|\underline{n}|} \times T_{G_f} \times U(1)^2$. To compute the instanton partition function for quiver gauge theories, we appeal to equivariant localization with respect to the complexification T of this larger torus. The equivariant parameters $(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon}) \in \mathfrak{t}$ were introduced in Section 3.3.

The flavor group G_f acts trivially on the moduli space $\mathfrak{M}(\underline{n}, \underline{k})$ and the T-action doesn't mix the different factors in (41). Thus, the problem of finding the T-fixed points simplifies to finding the fixed points for each factor $\mathfrak{M}(n_i, k_i)$, a problem we already solved in Section 3.2. By Theorem 3.11, we find that the fixed points for the quiver gauge theory are given by $\underline{n}$-colored partitions

$$\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^i)_{i \in \mathrm{Vert}_\gamma},$$

so for each vertex (factor $U(n_i)$ of the gauge group) there is a vector $\boldsymbol{\lambda}^i = (\lambda^{i\alpha})_{\alpha=1,\ldots,n_i}$ of n_i partitions whose sizes sum to k_i.

For each vertex i we define a T-equivariant matter bundle $\mathbf{M}_i \to \mathfrak{M}(\underline{n}, \underline{k})$. Topologically, it is trivial of rank m_i, with T acting on the total space as

$$e^{(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})} \cdot (x, v) \coloneqq (e^{(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})} \cdot x, e^{\mathfrak{m}_i} v).$$

Note that $e^{\mathfrak{m}_i}$ is an $m_i \times m_i$ matrix, so this makes sense. Alternatively, $\mathbf{M}_i$ may be seen as the pullback by the collapsing map $\mathfrak{M}(\underline{n}, \underline{k}) \to *$ of the representation $e^{(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})} \mapsto e^{\mathfrak{m}_i}$ of T. Similarly we define for each edge e a rank-1 T-equivariant matter bundle $\mathbf{M}_e \to \mathfrak{M}(\underline{n}, \underline{k})$ as the pullback of the representation $e^{(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})} \mapsto e^{\mathfrak{m}_e}$, by the collapsing map.

The bundles $\mathbf{K}, \mathbf{N} \to \mathfrak{M}(n, k)$ from remark 3.14 have relatives over $\mathfrak{M}(\underline{n}, \underline{k})$: For each vertex $i \in \mathrm{Vert}_\gamma$ we have a projection $\mathfrak{M}(\underline{n}, \underline{k}) \to \mathfrak{M}(n_i, k_i)$ to the i-th factor of the moduli space. To obtain vector bundles $\mathbf{K}_i, \mathbf{N}_i \to \mathfrak{M}(\underline{n}, \underline{k})$ we just pull back the bundles $\mathbf{K}, \mathbf{N}$ defined over $\mathfrak{M}(n_i, k_i)$ in remark 3.14. Armed with all of these bundles, following [Nek16] we define the virtual T-equivariant bundle

$$\mathbf{B} = \mathbf{B}^{\gamma, \underline{n}, \underline{m}} \coloneqq \bigoplus_{i \in \mathrm{Vert}_\gamma} \mathbf{M}_i^* \mathbf{K}_i \oplus \bigoplus_{e \in \mathrm{Edges}_\gamma} \mathbf{M}_e \left(\mathbf{N}_{\mathrm{out}(e)} \mathbf{K}^*_{\mathrm{in}(e)} \oplus (\wedge^2 \mathbf{Q}) \mathbf{N}^*_{\mathrm{in}(e)} \mathbf{K}_{\mathrm{out}(e)} \ominus (\wedge_{-1} \mathbf{Q}) \mathbf{K}_{\mathrm{out}(e)} \mathbf{K}^*_{\mathrm{in}(e)} \right) \tag{42}$$

of the quiver gauge theory $(\gamma, \underline{n}, \underline{m})$, where we suppress tensor products in our notation and the bundle $\mathbf{Q}$ is the pullback of the representation Q from Subsection 3.2.2 by the collapsing map $\pi : \mathfrak{M}(\underline{n}, \underline{k}) \to *$.

Then the instanton partition function of the quiver gauge theory is defined in terms of the bundle $\mathbf{B}$,

$$\mathcal{Z}_{\mathrm{inst}} \coloneqq \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \int_{\mathfrak{M}(\underline{n}, \underline{k})} \epsilon^T(\mathbf{B}), \tag{43}$$

where the sum is over instanton charges $\underline{k} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ and we define $\underline{\mathfrak{q}}^{\underline{k}} \coloneqq \prod_{i \in \mathrm{Vert}_\gamma} \mathfrak{q}_i^{k_i}$. Using equivariant localization, this is (formally) transformed to

$$\mathcal{Z}_{\mathrm{inst}} = \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \sum_{\underline{\boldsymbol{\lambda}} \in \mathfrak{P}(\underline{n}, \underline{k})} \frac{\epsilon^T(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}})}{\epsilon^T(T_{\underline{\boldsymbol{\lambda}}} \mathfrak{M}(\underline{n}, \underline{k}))}.$$

As $T_{\underline{\boldsymbol{\lambda}}} \mathfrak{M}(\underline{n}, \underline{k}) = \bigoplus_{i \in \mathrm{Vert}_\gamma} T_{\boldsymbol{\lambda}^i} \mathfrak{M}(n_i, k_i)$, the Euler class of the tangent representation is just the product of the Euler classes calculated in Section 2.5 (cf. (36)):

$$\epsilon^T(T_{\underline{\boldsymbol{\lambda}}} \mathfrak{M}(\underline{n}, \underline{k})) = \prod_{i \in \mathrm{Vert}_\gamma} \epsilon^T(T_{\boldsymbol{\lambda}^i} \mathfrak{M}(n_i, k_i))$$

The Euler class of the virtual representation $\mathbf{B}|_{\underline{\boldsymbol{\lambda}}}$ is defined as the quotient of the Euler classes of the positive and negative summands in (42). We list the characters of the representations involved:

$$\mathrm{Ch}^T(\mathbf{N}_i|_{\underline{\boldsymbol{\lambda}}}) = \sum_{\alpha=1}^{n_i} e^{\mathfrak{a}_{i\alpha}}, \qquad \mathrm{Ch}^T(\mathbf{K}_i|_{\underline{\boldsymbol{\lambda}}}) = \sum_{(\alpha,\square)\in\boldsymbol{\lambda}^i} e^{\phi_{i\alpha\square}}$$
$$\mathrm{Ch}^T(\wedge^2\mathbf{Q}|_{\underline{\boldsymbol{\lambda}}}) = e^{\varepsilon}, \qquad \mathrm{Ch}^T(\wedge_{-1}\mathbf{Q}|_{\underline{\boldsymbol{\lambda}}}) = 1 - e^{\varepsilon_1} - e^{\varepsilon_2} + e^{\varepsilon},$$
$$\mathrm{Ch}^T(\mathbf{M}_i|_{\underline{\boldsymbol{\lambda}}}) = \sum_{f=1}^{m_i} e^{\mathfrak{m}_{i,f}}, \qquad \mathrm{Ch}^T(\mathbf{M}_e|_{\underline{\boldsymbol{\lambda}}}) = e^{\mathfrak{m}_e}$$

The weights of the representations are the exponents in their characters. From this:

$$\begin{aligned}\epsilon^T(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}}) = &\prod_{i\in\mathrm{Vert}_\gamma}\left[\prod_{f=1}^{m_i}\prod_{(\alpha,\square)\in\boldsymbol{\lambda}^i}(\phi_{i\alpha\square}-\mathfrak{m}_{i,f})\right]\prod_{e\in\mathrm{Edges}_\gamma}\left[\prod_{(\alpha,\square)\in\boldsymbol{\lambda}^{\mathrm{in}(e)}}\prod_{\beta=1}^{n_{\mathrm{out}(e)}}(\mathfrak{m}_e+\mathfrak{a}_{\mathrm{out}(e),\beta}-\phi_{\mathrm{in}(e),\alpha,\square})\right]\\ &\prod_{e\in\mathrm{Edges}_\gamma}\left[\prod_{\alpha=1}^{n_{\mathrm{in}(e)}}\prod_{(\beta,\square)\in\boldsymbol{\lambda}^{\mathrm{out}(e)}}(\mathfrak{m}_e-\mathfrak{a}_{\mathrm{in}(e),\alpha}+\phi_{\mathrm{out}(e),\beta,\square}+\varepsilon)\right]\\ &\prod_{e\in\mathrm{Edges}_\gamma}\left[\prod_{(\alpha,\square)\in\boldsymbol{\lambda}^{\mathrm{in}(e)}}\prod_{(\beta,\square')\in\boldsymbol{\lambda}^{\mathrm{out}(e)}}S(\mathfrak{m}_e-\phi_{\mathrm{in}(e),\alpha,\square}+\phi_{\mathrm{out}(e),\beta,\square'})\right]\end{aligned} \tag{44}$$

where $S(x) := \frac{1}{c_x^T(\wedge_{-1}Q)} = \frac{(x+\varepsilon_1)(x+\varepsilon_2)}{x(x+\varepsilon)}$. Ultimately, the instanton partition function of the quiver gauge theory is

$$\mathcal{Z}_{\mathrm{inst}} = \sum_{\underline{\boldsymbol{\lambda}}\in\mathfrak{P}(\underline{n})} \underline{\mathfrak{q}}^{\underline{k}[\boldsymbol{\lambda}]} Z_{\underline{\boldsymbol{\lambda}}}, \qquad \text{where} \qquad Z_{\underline{\boldsymbol{\lambda}}} = \frac{\epsilon^T(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}})}{\epsilon^T(T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n},\underline{k}))}. \tag{45}$$

Localization in K-theory. The K-theory instanton partition function of $(\gamma, \underline{n}, \underline{m})$ is

$$\mathcal{Z}^K_{\mathrm{inst}} = \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \pi_*\Big(\epsilon^T\big(\mathbf{B} \to \mathfrak{M}(\underline{n},\underline{k})\big)\Big) \tag{46}$$

where ϵ^T denotes the Euler class in equivariant K-theory rather than in cohomology, and $\pi_* : K_T(\mathfrak{M}(\underline{n},\underline{k})) \to K_T(*) \cong \mathbb{C}[T]$ is induced by the collapsing map. Formally applying Theorem 2.19, we get

$$\mathcal{Z}^K_{\mathrm{inst}} = \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \sum_{\underline{\boldsymbol{\lambda}}\in\mathfrak{P}(\underline{n},\underline{k})} \frac{\epsilon^T(\mathbf{B}|_{\underline{\boldsymbol{\lambda}}})}{\epsilon^T(T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n},\underline{k}))} = \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \sum_{\underline{\boldsymbol{\lambda}}\in\mathfrak{P}(\underline{n},\underline{k})} Z_{\underline{\boldsymbol{\lambda}}}.$$

To give an explicit expression of $Z^K_{\underline{\boldsymbol{\lambda}}}$, one just translates (44) into K-theory by the prescription $\prod X \to \prod(1-e^{-X})$.

4 Non-perturbative Dyson–Schwinger equations and qq-characters

Let us start off by summarizing the probabilistic setup given by the instanton partition function (45). Suppose given a quiver gauge theory $(\gamma, \underline{n}, \underline{m})$, consisting of a quiver γ, a coloring $\underline{n} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$ determining the gauge group, and a coloring $\underline{m} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ determining the matter content. The discrete set of $\underline{n}$-colored partitions (of all sizes) is denoted by $\mathfrak{P}(\underline{n})$. The $\mu_{\underline{\boldsymbol{\lambda}}}$ occuring in the partition function $\mathcal{Z}_{\mathrm{inst}} = \sum_{\underline{\boldsymbol{\lambda}} \in \mathfrak{P}(\underline{n})} \mu_{\underline{\boldsymbol{\lambda}}}$ are rational functions on the Lie algebra $\mathfrak{t}$, whose elements we denote by $(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})$. For each generic choice of equivariant parameters $(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})$, we think of

$$\mu(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon}) : \mathfrak{P}(\underline{n}) \to \mathbb{C}, \quad \underline{\boldsymbol{\lambda}} \mapsto \mu_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})$$

as the mass function of a complex measure on $\mathfrak{P}(\underline{n})$. We assume that the measure's total mass (i.e. $\mathcal{Z}_{\mathrm{inst}}(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})$) is finite, thus entering the realm of (complex) probability theory. In this setting, an observable $\mathcal{O}$ is a function $\mathfrak{P}(\underline{n}) \to V$ valued in some complex vector space ($\mathbb{C}$ or the space of rational functions on $\mathbb{C}$). Its expectation is then (we do not normalize)

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon}} = \sum_{\underline{\boldsymbol{\lambda}} \in \mathfrak{P}(\underline{n})} \mu_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})\, O[\underline{\boldsymbol{\lambda}}] \tag{47}$$

which may also, by treating $(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon})$ as variable, be interpreted as a function $\mathfrak{t} \to V$ (defined almost everywhere).

Our objective in this section is, in a sense, to investigate the measures μ, for example their limit properties for $\vec{\varepsilon} \to 0$. To do this, we make use of the non-perturbative Dyson–Schwinger equations introduced by Nekrasov in [Nek16]. These are based on a kind of discrete symmetry of the measure which causes that the expectations of certain observables, called qq-characters, valued in the vector space of rational functions on $\mathbb{C}$, are free of poles. These qq-characters are determined entirely by the quiver γ and its colorings $\underline{n}, \underline{m}$. They connect to the representation theory of quantum groups associated to the quiver, as we will outline in Section 4.5. Simple qq-characters will be computed recursively, starting off with a (or a product of) $\mathcal{Y}$-observable, also rational function-valued. The expectation of this random rational function initially has poles which we cancel by adding a product of $\mathcal{Y}$-observables and their inverses. This may in turn introduce new poles to the expectation which are again cancelled by adding new $\mathcal{Y}$-observables in the same way. This process of recursive pole cancellation produces qq-characters as Laurent series in the $\mathcal{Y}$-observables.

4.1 Non-perturbative transformation of the measure

The measure on the set $\mathfrak{P}(\underline{n})$ of $\underline{n}$-colored partitions is given by

$$\mu_{\underline{\boldsymbol{\lambda}}} = \underline{\mathfrak{q}}^{\underline{k}[\underline{\boldsymbol{\lambda}}]} Z_{\underline{\boldsymbol{\lambda}}} = \underline{\mathfrak{q}}^{\underline{k}[\underline{\boldsymbol{\lambda}}]} \epsilon^T(\mathbf{B}_{\underline{\boldsymbol{\lambda}}} - T_{\underline{\boldsymbol{\lambda}}} \mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}])), \tag{48}$$

a rational function of $\underline{\mathfrak{a}}, \underline{\mathfrak{m}}, \vec{\varepsilon}$. The non-perturbative transformations on $\mathfrak{P}(\underline{n})$ consist of changing a $\underline{\boldsymbol{\lambda}}$ by adding (or removing) a box. Our $\underline{n}$-colored partition $\underline{\boldsymbol{\lambda}}$ really consists of many partitions; $\underline{\boldsymbol{\lambda}} = (\lambda^{i,\alpha})_{i \in \mathrm{Vert}_\gamma, \alpha = 1, \ldots, n_i}$. Thus, to add a box to $\underline{\boldsymbol{\lambda}}$, we need to specify at which vertex $i_+ \in \mathrm{Vert}_\gamma$ and for which index $\alpha_+ = 1, \ldots, n_{i_+}$ we are changing the component partition λ^{i_+, α_+}. We can then add a box to λ^{i_+, α_+}:

$$\lambda^{i_+, \alpha_+} \to \lambda_+^{i_+, \alpha_+} = \lambda^{i_+, \alpha_+} \cup \square_+$$

where $\square_+ \in \partial_+ \lambda^{i_+,\alpha_+}$ must be in the outer boundary (c.f. appendix A). This changes the $\underline{n}$-colored partition, and we write

$$\underline{\boldsymbol{\lambda}} \to \underline{\boldsymbol{\lambda}}_+ = \underline{\boldsymbol{\lambda}} \cup (i_+, \alpha_+, \square_+) \qquad \text{where} \qquad (i_+, \alpha_+, \square_+) \in \partial_+ \underline{\boldsymbol{\lambda}}.$$

Clearly, adding a box increases the size of λ^{i_+,α_+} by one: $k_i[\underline{\boldsymbol{\lambda}}_+] = k_i[\underline{\boldsymbol{\lambda}}] + \delta_{i,i_+}$.

The reason for considering these non-perturbative transformations is that they change the measure (48) in a predictable way. We will calculate this:

$$\frac{Z_{\underline{\boldsymbol{\lambda}}_+}}{Z_{\underline{\boldsymbol{\lambda}}}} = \frac{\epsilon^T\Big(\mathbf{B}_{\underline{\boldsymbol{\lambda}}_+} - T_{\underline{\boldsymbol{\lambda}}_+}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}_+])\Big)}{\epsilon^T\Big(\mathbf{B}_{\underline{\boldsymbol{\lambda}}} - T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}])\Big)} = \epsilon^T\Big(\mathbf{B}_{\underline{\boldsymbol{\lambda}}_+} - T_{\underline{\boldsymbol{\lambda}}_+}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}_+]) - \mathbf{B}_{\underline{\boldsymbol{\lambda}}} + T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}])\Big) \tag{49}$$

We write $x_+ = \phi_{i_+,\alpha_+,\square_+}$, and define the virtual representations

$$S_i[\underline{\boldsymbol{\lambda}}] \coloneqq N_i - (\wedge_{-1}Q)K_i[\underline{\boldsymbol{\lambda}}],$$

where $K_i[\underline{\boldsymbol{\lambda}}]$ is the isotropy representation $\iota^*_{\underline{\boldsymbol{\lambda}}}\mathbf{K}_i$.

Lemma 4.1. *The virtual representation*

$$\mathbf{B}_{\underline{\boldsymbol{\lambda}}_+} - T_{\underline{\boldsymbol{\lambda}}_+}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}_+]) - \mathbf{B}_{\underline{\boldsymbol{\lambda}}} + T_{\underline{\boldsymbol{\lambda}}}\mathfrak{M}(\underline{n}, \underline{k}[\underline{\boldsymbol{\lambda}}])$$

is isomorphic to

$$\begin{aligned} &-e^{-x_+}S_{i_+}[\underline{\boldsymbol{\lambda}}] - e^{x_++\varepsilon}S^*_{i_+}[\underline{\boldsymbol{\lambda}}_+] + e^{x_+}M^*_{i_+} \\ &\qquad + \sum_{e:s\to i_+} e^{x_++\varepsilon}e^{\mathfrak{m}_e}S^*_s[\underline{\boldsymbol{\lambda}}] + \sum_{e:i_+\to t} e^{-x_+}e^{\mathfrak{m}_e}S_t[\underline{\boldsymbol{\lambda}}] - \sum_{e:i_+\to i_+} e^{\mathfrak{m}_e}(\wedge_{-1}Q). \end{aligned} \tag{50}$$

Proof. A computation using (34), (42), $K_i[\underline{\boldsymbol{\lambda}}_+] - K_i[\underline{\boldsymbol{\lambda}}] \cong \delta_{i,i_+}e^{x_+}$, and $e^{-\varepsilon}(\wedge_{-1}Q) \cong \wedge_{-1}Q^*$. □

Here we denote by e^x the one-dimensional representation of T whose character is e^x. In particular, $e^\varepsilon \cong \wedge^2 Q$. We want to evaluate the equivariant Euler classes of the summand representations in (50). To do this, note that for a one-dimensional representation e^x and a (virtual) representation R,

$$\epsilon^T(e^x R) = c^T_x(R)$$

is exactly the equivariant Chern polynomial of the representation, evaluated at x. One sees this by observing that the weights of $e^x R$ are exactly the sums $x + w$ where w is some weight of R. Meanwhile, the Chern polynomial is the product of terms $x + w$ for weights w of R.

Definition 4.2. *The i-th $\mathcal{Y}$-observable is the equivariant Chern polynomial (really a rational function) of the virtual representation $S_i[\underline{\boldsymbol{\lambda}}]$:*

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}] \coloneqq c^T_x(S_i[\underline{\boldsymbol{\lambda}}]^*) \tag{51}$$

We also define the i-th matter polynomial and the S-function:

$$P_i(x) \coloneqq c^T_x(M^*_i) = \prod_{f=1}^{m_i}(x - \mathfrak{m}_{i,f}), \qquad S(x) \coloneqq \frac{1}{c^T_x(\wedge_{-1}Q)} = \frac{(x+\varepsilon_1)(x+\varepsilon_2)}{x(x+\varepsilon)}$$

$\mathcal{Y}_i(x)$ depends on the random $\underline{n}$-colored partition $\underline{\boldsymbol{\lambda}}$, hence it is a random variable valued in the vector space of rational functions on $\mathbb{C}$. Then

$$\epsilon^T(e^x S_i^*[\underline{\boldsymbol{\lambda}}]) = \mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}], \qquad \epsilon^T(e^x M_i^*) = P_i(x), \qquad \epsilon^T(e^{\mathfrak{m}_e}(\wedge_{-1}Q)) = S(\mathfrak{m}_e)^{-1},$$

with which we can evaluate the quotient (49):

$$\begin{aligned}
\frac{\mu_{\underline{\boldsymbol{\lambda}}_+}}{\mu_{\underline{\boldsymbol{\lambda}}}} &= \mathfrak{q}_{i_+}\frac{Z_{\underline{\boldsymbol{\lambda}}_+}}{Z_{\underline{\boldsymbol{\lambda}}}} = \mathfrak{q}_{i_+}\epsilon^T(\text{expression (50)}) \\
&= \mathfrak{q}_{i_+}P_{i_+}(x_+)\frac{\prod_{e:s\to i_+}\mathcal{Y}_s(x_+ + \varepsilon + \mathfrak{m}_e)\prod_{e:i_+\to t}(-1)^{\dim S_t[\underline{\boldsymbol{\lambda}}]}\mathcal{Y}_t(x_+ - \mathfrak{m}_e)}{(-1)^{\dim S_{i_+}[\underline{\boldsymbol{\lambda}}]}\mathcal{Y}_{i_+}(x_+)[\underline{\boldsymbol{\lambda}}]\mathcal{Y}_{i_+}(x_+ + \varepsilon)[\underline{\boldsymbol{\lambda}}_+]}\prod_{e:i_+\to i_+}S(\mathfrak{m}_e) \\
&= (-1)^{\kappa_{i_+}}\mathfrak{q}_{i_+}P_{i_+}(x_+)\frac{\prod_{e:s\to i_+}\mathcal{Y}_s(x_+ + \varepsilon + \mathfrak{m}_e)\prod_{e:i_+\to t}\mathcal{Y}_t(x_+ - \mathfrak{m}_e)}{\mathcal{Y}_{i_+}(x_+)[\underline{\boldsymbol{\lambda}}]\mathcal{Y}_{i_+}(x_+ + \varepsilon)[\underline{\boldsymbol{\lambda}}_+]}\prod_{e:i_+\to i_+}S(\mathfrak{m}_e),
\end{aligned} \tag{52}$$

where the RHS should be interpreted as the limit $x \longrightarrow x_+$ and

$$\begin{aligned}
\kappa_{i_+} :&= -\dim S_{i_+}[\underline{\boldsymbol{\lambda}}] + \sum_{e:i_+\to t}\dim S_t[\underline{\boldsymbol{\lambda}}] = -n_{i_+} + 2k_{i_+} - 2k_{i_+} + \sum_{e:i_+\to t}(n_t + 2k_t - 2k_t) \\
&= -n_{i_+} + \sum_{e:i_+\to t} n_t.
\end{aligned} \tag{53}$$

We see that the transformation of the measure under addition of a box is given by the $\mathcal{Y}$-observables $\mathcal{Y}_i(x)$, one for each vertex $i \in \mathrm{Vert}_\gamma$. We point out that the function $\mathcal{Y}_{i_+}(x_+)[\underline{\boldsymbol{\lambda}}]$ in the denominator of (52) has a zero at x_+ (this is apparent from (55)), but this is cancelled by $\mathcal{Y}_{i_+}(x_+ + \varepsilon)[\underline{\boldsymbol{\lambda}}_+]$ which has a pole there. This is why the RHS in (52) should rather be interpreted as the limit $x \longrightarrow x_+$.

The $\mathcal{Y}_i$-observables in terms of partitions. Let us examine the $\mathcal{Y}_i$-observables more closely. We defined them in (51) as Chern polynomials (really rational functions as the representations are virtual)

$$\mathcal{Y}_i(x) = c_x^T(N_i^* - \wedge_{-1}Q^*K_i[\underline{\boldsymbol{\lambda}}]^*) = \frac{c_x^T(N_i^*)}{c_x^T(\wedge_{-1}Q^*K_i[\underline{\boldsymbol{\lambda}}]^*)}.$$

Using the weights of the various representations involved, we rewrite this

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}] = \prod_{\alpha=1}^{n_i}(x - \mathfrak{a}_{i,\alpha})\prod_{\square\in\lambda^{i,\alpha}}S(-x + \phi_{i,\alpha,\square}) \tag{54}$$

where we use the S-function $S(x) = c_x^T(\wedge_{-1}Q)^{-1} = c_{-x}^T(\wedge_{-1}Q^*)^{-1} = \frac{(x+\varepsilon_1)(x+\varepsilon_2)}{x(x+\varepsilon)}$ which is the reciprocal of the Chern polynomial (actually a rational function as $\wedge_{-1}Q$ is virtual) of $\wedge_{-1}Q$. The second product in (54) is telescopic, simplifying to

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}] = \prod_{\alpha=1}^{n_i}\frac{\prod_{\square\in\partial_+\lambda^{(i,\alpha)}}(x - \phi_{i,\alpha,\square})}{\prod_{\square\in\partial_-\lambda^{(i,\alpha)}}(x - \phi_{i,\alpha,\square} - \varepsilon)}, \tag{55}$$

which expresses the $\mathcal{Y}_i$-observable in terms of the inner and outer boundaries of $\underline{\boldsymbol{\lambda}}$. As the outer boundary contains one more box than the inner boundary (cf. appendix A), the degree of the rational function $\mathcal{Y}_i(x)$

is n_i. In the form (55), the zeros (resp. poles) of $\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}]$ are identified with the boxes in the outer (resp. inner) boundary. Note that when $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$, then $\mathcal{Y}_i(x) = \prod_\alpha (x - \mathfrak{a}_{i,\alpha})$ is independent of $\underline{\boldsymbol{\lambda}}$, i.e. deterministic. The $\mathcal{Y}_i$-observables are physically interpreted as the characteristic polynomials of the adjoint Higgs fields Φ_i.

4.2 Non-perturbative Dyson–Schwinger equations

In the previous section we gave an expression for the transformation law of the measure $\mu_{\underline{\boldsymbol{\lambda}}}$ under the addition of a box;

$$\underline{\boldsymbol{\lambda}}_+ = \underline{\boldsymbol{\lambda}} \cup (i_+, \alpha_+, \square_+) \quad \text{where} \quad (i_+, \alpha_+, \square_+) \in \partial_+ \underline{\boldsymbol{\lambda}}.$$

We use the notations

$$x_+ = \phi_{i_+, \alpha_+, \square_+}, \qquad \phi_{i,\alpha,\square} = \mathfrak{a}_{i\alpha} + (s_1 - 1)\varepsilon_1 + (s_2 - 1)\varepsilon_2 \quad (\text{for } \square = (s_1, s_2)).$$

We start by examining the simplest quiver, A_1, consisting of one vertex and no edges. In this case, equation (52) becomes

$$\frac{\mu_{\boldsymbol{\lambda}_+}}{\mu_{\boldsymbol{\lambda}}} = (-1)^\kappa \mathfrak{q} \lim_{x \to x_+} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]\mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}_+]}.$$

For the residue at x_+, this implies

$$\mu_{\boldsymbol{\lambda}_+} \operatorname*{Res}_{x=x_+} \mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}_+] = (-1)^\kappa \mu_{\boldsymbol{\lambda}} \mathfrak{q} \operatorname*{Res}_{x=x_+} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]} \tag{56}$$

(as $\square_+ \in \partial_- \lambda_+^{\alpha_+}$ and $\square_+ \in \partial_+ \lambda^{\alpha_+}$, both functions have a pole at x_+). From now on, we choose to ignore the sign $(-1)^\kappa$ by redefining $\mathfrak{q}$ as $(-1)^{\kappa+1}\mathfrak{q}$, which is standard practice for qq-characters. From equation (56), we derive the fundamental non-perturbative Dyson–Schwinger equation of A_1 quiver gauge theory:

Proposition 4.3. *In the quiver gauge theory (A_1, n, m) ($n \geq 1$ and $m \geq 0$ are arbitrary), the expectation*

$$\left\langle \mathcal{Y}(x+\varepsilon) + \mathfrak{q}\frac{P(x)}{\mathcal{Y}(x)} \right\rangle \tag{57}$$

is a polynomial of degree $\max(n, m-n)$.

Proof. Equation (55) makes it clear that the poles of $\mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}]$ are exactly at $\phi_{\alpha\square}$ for $(\alpha, \square) \in \partial_- \boldsymbol{\lambda}$, i.e. they correspond to the inner boundary boxes. Similarly, the poles of $\mathfrak{q}P(x)/\mathcal{Y}(x)[\boldsymbol{\lambda}]$ are the $\phi_{\alpha\square}$ for $(\alpha, \square)$ in the outer boundary $\partial_+ \boldsymbol{\lambda}$.

Furthermore, from (55) one sees that the set of equivariant parameters $(\mathfrak{a}, \vec{\varepsilon}) \in \mathfrak{t}$ for which $\mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}]$ or $\frac{1}{\mathcal{Y}(x)[\boldsymbol{\lambda}]}$ have only single poles is generic. Thus, we may restrict to this case.

For a box $(\alpha, \square) \in \{1, \ldots, n\} \times \mathbb{Z}^2_{>0}$, we define the sets

$$\partial_-^{-1}(\alpha, \square) \coloneqq \{\boldsymbol{\lambda} \in \mathfrak{P}(n) \mid (\alpha, \square) \in \partial_- \boldsymbol{\lambda}\}, \qquad \partial_+^{-1}(\alpha, \square) \coloneqq \{\boldsymbol{\lambda} \in \mathfrak{P}(n) \mid (\alpha, \square) \in \partial_+ \boldsymbol{\lambda}\}.$$

Noting that we add boxes to Young diagrams by turning outer boundaries into inner boundaries we have a bijection

$$\partial_+^{-1}(\alpha, \square) \leftrightarrow \partial_-^{-1}(\alpha, \square), \quad \boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cup (\alpha, \square).$$

Using the definition of the expectation in (47), we write the expectation (57) as an infinite sum which we assume to converge uniformly on compact sets:

$$\left\langle \mathcal{Y}(x+\varepsilon) + \mathfrak{q}\frac{P(x)}{\mathcal{Y}(x)} \right\rangle = \sum_{\boldsymbol{\lambda}\in\mathfrak{P}(n)} \mu_{\boldsymbol{\lambda}} \mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}] + \sum_{\boldsymbol{\lambda}\in\mathfrak{P}(n)} \mu_{\boldsymbol{\lambda}} \mathfrak{q} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]} \tag{58}$$

If $p \in \mathbb{C}$ is a pole of (58), then it must be a pole of one of the summands, i.e. there is a $\boldsymbol{\lambda} \in \mathfrak{P}(n)$ such that $p = \phi_{\alpha\square}$ for some $(\alpha,\square) \in \partial_+\boldsymbol{\lambda}$ or $\partial_-\boldsymbol{\lambda}$. The partitions in the sum (58) which contribute a pole at p are exactly $\boldsymbol{\lambda} \in \partial_-^{-1}(\alpha,\square)$ for $\mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}]$ and $\boldsymbol{\lambda} \in \partial_+^{-1}(\alpha,\square)$ for $\frac{1}{\mathcal{Y}(x)[\boldsymbol{\lambda}]}$. Thus, the part of the sum (58) that contributes poles at p is

$$\sum_{\boldsymbol{\lambda}\in\partial_-^{-1}(\alpha,\square)} \mu_{\boldsymbol{\lambda}} \mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}] + \sum_{\boldsymbol{\lambda}\in\partial_+^{-1}(\alpha,\square)} \mu_{\boldsymbol{\lambda}} \mathfrak{q} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]} \tag{59}$$

Using the bijection $\partial_+^{-1}(\alpha,\square) \cong \partial_-^{-1}(\alpha,\square)$, we rewrite this

$$\sum_{\boldsymbol{\lambda}\in\partial_+^{-1}(\alpha,\square)} \left(\mu_{\boldsymbol{\lambda}\cup(\alpha,\square)} \mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}\cup(\alpha,\square)] + \mu_{\boldsymbol{\lambda}} \mathfrak{q} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]} \right) \tag{60}$$

But, by (56), we know that at p,

$$\operatorname*{Res}_{x=p} \left(\mu_{\boldsymbol{\lambda}\cup(\alpha,\square)} \mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}\cup(\alpha,\square)] + \mu_{\boldsymbol{\lambda}} \mathfrak{q} \frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]} \right) = 0,$$

so that the sum (59) has residue zero at p, meaning that (58) has no pole there. To exchange Res and the infinite sum in (60), we use uniform convergence of the series on some circle around p.

Now that we know there are no poles in expectation, what's left to check is that the expectation is a polynomial. Observing that $\mathcal{Y}(x)[\boldsymbol{\lambda}]$ is a rational function whose numerator polynomial has degree n higher than the denominator polynomial, the following lemma closes the proof. $\square$

Lemma 4.4. *Suppose P_k, Q_k, $k \in \mathbb{Z}_{>0}$ are polynomials $\mathbb{C} \to \mathbb{C}$ such that* $\deg(P_k) - \deg(Q_k) \le n \ge 0$ *for all k and that, away from the zeros of $Q_1, Q_2, \ldots$, the series*

$$\sum_{k=1}^{\infty} \frac{P_k}{Q_k}$$

converges uniformly on compact sets. Then, if the limit is an entire function, it is a polynomial of degree at most n.

Proof. Using polynomial division,

$$\frac{P_k(x)}{Q_k(x)} = F_k(x) + R_k(x),$$

where F_k is a polynomial of degree at most n and R_k is a rational function whose numerator has strictly lower degree than the denominator.

F_k is the regular part of the Laurent expansion of P_k/Q_k around infinity, thus $\sum_k F_k$ is the regular part of the Laurent expansion of $\sum_k P_k/Q_k$ around infinity and in particular converges and then so does

$\sum_k R_k$. Then $\sum_k F_k$ is a polynomial of degree at most n and $\lim_{x\to\infty} \sum_k R_k(x) = 0$. Thus,

$$\lim_{x\to\infty} \frac{\sum_k P_k(x)/Q_k(x)}{\sum_k F_k(x)} = 1,$$

meaning that $\sum_k P_k/Q_k$ grows like a polynomial. By the extended Liouville theorem, an entire function that grows no faster than $|x|^n$ must be a polynomial of degree at most n. □

Note that when the quiver gauge theory in Proposition 4.3 has β function $\beta \leq 0$, then the expression (40) of β shows that the degree of the expectation is n.

The expression $\mathcal{X}_{\underline{\delta}_1,0}(x) = \mathcal{Y}(x+\varepsilon) + \mathfrak{q}\frac{P(x)}{\mathcal{Y}(x)}$ is called the *fundamental qq-character* of the quiver gauge theory (A_1, n, m). The non-perturbative Dyson–Schwinger equation is

$$\langle \mathcal{X}_{\underline{\delta}_1,0}(x)\rangle = \text{a polynomial of degree } n,$$

saying that the qq-character's expectation is a polynomial. We know little about this polynomial apart from its degree. From (58) it can be seen that its leading coefficient must be $\mathcal{Z}_{\text{inst}}$ in the case $\beta < 0$, $\mathcal{Z}_{\text{inst}} + \mathfrak{q}\mathcal{Z}_{\text{inst}}$ in the case $\beta = 0$, and $\mathfrak{q}\mathcal{Z}_{\text{inst}}$ in the case $\beta > 0$.

Our goal is to define a qq-character for any quiver gauge theory $(\gamma, \underline{n}, \underline{m})$, always such that their expectations are polynomial, i.e. that they satisfy a non-perturbative Dyson–Schwinger equation. The idea is to start with some monomial of $\mathcal{Y}_i$-functions

$$\prod_{i\in\mathrm{Vert}_\gamma} \prod_{p=1}^{w_i} \mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})$$

where $\underline{w} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ is the *weight* of the qq-character and $\underline{\sigma} = (\sigma_{i,p})_{i\in\mathrm{Vert}_\gamma, p=1,\dots,w_i} \in \mathbb{C}^{\mathrm{Vert}_\gamma}$ is a collection of shift parameters. This initial monomial is called the qq-character's *highest weight monomial* in reference to connections with highest weight representations of simple Lie algebras and quantum groups which we will discuss in Section 4.5. The expectation of the highest weight monomial has singularities in the complex variable x. Using the non-perturbative transformation of the measure (52), the singularities can be removed by adding another Laurent monomial in the $\mathcal{Y}_i$-functions, constructed using the procedure of *iWeyl reflection* we will demonstrate for other examples. The monomial added by iWeyl reflection may introduce new singularities, which are removed using the same procedure. This process may never terminate, as we will see for the example of the $\hat{A}_0$ quiver. The general principle (no proof is known to us) is that this process's end result (or its limit for infinitely many steps), which is called the qq-character $\mathcal{X}_{\underline{w},\underline{\sigma}}(x)$, has non-singular expectation, and in the case where $\underline{\beta} \leq 0$ the expectation is polynomial of degree $\underline{w}\cdot\underline{n}$.

In this section the term qq-character will remain without clear definition; referring to a function expressed in terms of $\mathcal{Y}_i$-functions whose expectation satisfies a non-perturbative Dyson–Schwinger equation. In Section 4.4, we will give a geometric definition of qq-characters. By the i-th fundamental qq-character ($i \in \mathrm{Vert}_\gamma$), we mean the qq-character with weight $\underline{\delta}_i = (\delta_{i,j})_{j\in\mathrm{Vert}_\gamma}$ (e.g. for $\gamma = A_1$ in Proposition 4.3.)

A_2 quiver. We consider the quiver

$$1 \longrightarrow 2$$

with vertices $1, 2$, and edge e. In this case there are two $\mathcal{Y}$-observables, $\mathcal{Y}_1(x)$ and $\mathcal{Y}_2(x)$, as well as two matter polynomials $P_1(x)$, $P_2(x)$ and an edge mass $\mathfrak{m}_e$. The edge complicates the qq-character. Suppose that, to an $\underline{n}$-colored partition $\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2)$, we add a box $(1, \alpha_+, \square_+)$: $\underline{\boldsymbol{\lambda}}_+ = \underline{\boldsymbol{\lambda}} \cup (1, \alpha_+, \square_+)$. Then, by (52), the measure transforms

$$\frac{\mu_{\underline{\boldsymbol{\lambda}}_+}}{\mu_{\underline{\boldsymbol{\lambda}}}} = (-1)^{\kappa_1} \mathfrak{q}_1 \lim_{x \to x_+} \left(\frac{P_1(x)}{\mathcal{Y}_1(x)[\underline{\boldsymbol{\lambda}}]\mathcal{Y}_1(x+\varepsilon)[\underline{\boldsymbol{\lambda}}_+]} \mathcal{Y}_2(x - \mathfrak{m}_e) \right), \tag{61}$$

where $x_+ = \phi_{1,\alpha_+,\square_+}$. The proof of Proposition 4.3 is easily adapted to show that

$$\left\langle \mathcal{Y}_1(x+\varepsilon) + \mathfrak{q}_1 \frac{P_1(x)}{\mathcal{Y}_1(x)} \mathcal{Y}_2(x - \mathfrak{m}_e) \right\rangle \tag{62}$$

has no poles at $x = \phi_{1,\alpha,\square}$ for any box $(1, \alpha, \square)$, at least when $\mathcal{Y}_2(x - \mathfrak{m}_e)$ has no poles there, which is the case for generic $\underline{\mathfrak{a}}, \mathfrak{m}_e, \vec{\varepsilon}$. However, $\mathcal{Y}_2(x - \mathfrak{m}_e)$ introduces new singularities so that the above expression is not yet polynomial but needs further desingularization. Using the version of (61) for the second vertex, i.e.

$$\frac{\mu_{\underline{\boldsymbol{\lambda}}_+}}{\mu_{\underline{\boldsymbol{\lambda}}}} = (-1)^{\kappa_2} \mathfrak{q}_2 \lim_{x \to x_+} \left(\frac{P_2(x)}{\mathcal{Y}_2(x)[\underline{\boldsymbol{\lambda}}]\mathcal{Y}_2(x+\varepsilon)[\underline{\boldsymbol{\lambda}}_+]} \mathcal{Y}_1(x + \varepsilon + \mathfrak{m}_e) \right)$$

where $\underline{\boldsymbol{\lambda}}_+ = \underline{\boldsymbol{\lambda}} \cup (2, \alpha_+, \square_+)$ and $x_+ = \phi_{2,\alpha_+,\square_+}$, and applying the arguments from the proof of Proposition 4.3, one sees that

$$\left\langle \frac{1}{\mathcal{Y}_1(x)} \mathcal{Y}_2(x - \mathfrak{m}_e) + \frac{1}{\mathcal{Y}_1(x)} \mathfrak{q}_2 \frac{P_2(x - \mathfrak{m}_e - \varepsilon)}{\mathcal{Y}_2(x - \mathfrak{m}_e - \varepsilon)} \mathcal{Y}_1(x) \right\rangle \tag{63}$$

has no poles contributed by $\mathcal{Y}_2(x - \mathfrak{m}_e)$. Thus, combining (62) and (63),

$$\left\langle \mathcal{Y}_1(x+\varepsilon) + \mathfrak{q}_1 \frac{P_1(x)}{\mathcal{Y}_1(x)} \mathcal{Y}_2(x - \mathfrak{m}_e) + \mathfrak{q}_1\mathfrak{q}_2 \frac{P_1(x)P_2(x - \mathfrak{m}_e - \varepsilon)}{\mathcal{Y}_2(x - \mathfrak{m}_e - \varepsilon)} \right\rangle \tag{64}$$

has no poles at all. So this is the desingularization of $\langle \mathcal{Y}_1(x+\varepsilon) \rangle$ for the A_2 quiver. In fact, by Lemma 4.4, it is a polynomial of degree $\max(n_1, m_1 - n_1 + n_2, m_1 + m_2 - n_2)$. The same procedure, but starting with the other vertex's observable $\langle \mathcal{Y}_2(x+\varepsilon) \rangle$, yields that

$$\left\langle \mathcal{Y}_2(x+\varepsilon) + \mathfrak{q}_2 \frac{P_2(x)}{\mathcal{Y}_2(x)} \mathcal{Y}_1(x + \varepsilon + \mathfrak{m}_e) + \mathfrak{q}_2\mathfrak{q}_1 \frac{P_2(x)P_1(x + \mathfrak{m}_e)}{\mathcal{Y}_1(x + \mathfrak{m}_e)} \right\rangle \tag{65}$$

is a polynomial of degree $\max(n_2, m_2 - n_2 + n_1, m_2 + m_1 - n_1)$. The expressions inside the expectation brackets in (64) and (65) are the fundamental qq-characters (one for each vertex) of the A_2 quiver.

Proposition 4.5. *In the quiver gauge theory* $(1 \to 2, \underline{n}, \underline{m})$, *where* $\underline{n} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$ *and* $\underline{m} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ *are arbitrary colorings, the expectation* (64) *(resp.* (65)*) is polynomial of degree* $\max(n_1, m_1 - n_1 + n_2, m_1 + m_2 - n_2)$ *(resp. polynomial of degree* $\max(n_2, m_2 - n_2 + n_1, m_2 + m_1 - n_1)$*).*

Using (40) it is seen that when the β function is bounded $\beta_1, \beta_2 \leq 0$, the degree is n_1 (resp. n_2).

iWeyl reflection and a proposed algorithm for computing qq-characters. The qq-characters we have looked at so far have been generated by the pole cancelling operation

$$\mathrm{iWeyl} : \mathcal{Y}_i(x+\varepsilon) \to \mathcal{Y}_i(x+\varepsilon) \frac{\mathfrak{q}_i P_i(x)}{A_i(x)}, \tag{66}$$

where

$$A_i(x) \coloneqq \mathcal{Y}_i(x+\varepsilon)\mathcal{Y}_i(x)\left(\prod_{e:\mathrm{in}(e)=i}\mathcal{Y}_{\mathrm{out}(e)}(x-\mathfrak{m}_e)\prod_{e:\mathrm{out}(e)=i}\mathcal{Y}_{\mathrm{in}(e)}(x+\varepsilon+\mathfrak{m}_e)\prod_{e:\mathrm{in}(e)=\mathrm{out}(e)=i}S(\mathfrak{m}_e)\right)^{-1}.$$

The operation (66) is called iWeyl (''instanton'' Weyl) reflection, named so for analogies with the Weyl reflections of root systems of simple Lie algebras we will explore in Section 4.5. The iWeyl reflection satisfies that

$$\begin{aligned}&\left\langle \mathcal{Y}_i(x+\varepsilon)+\mathrm{iWeyl}(\mathcal{Y}_i(x+\varepsilon))\right\rangle\\ &=\left\langle \mathcal{Y}_i(x+\varepsilon)+\mathfrak{q}_iP_i(x)\frac{\prod_{e:\mathrm{in}(e)=i}\mathcal{Y}_{\mathrm{out}(e)}(x-\mathfrak{m}_e)\prod_{e:\mathrm{out}(e)=i}\mathcal{Y}_{\mathrm{in}(e)}(x+\varepsilon+\mathfrak{m}_e)}{\mathcal{Y}_i(x)}\prod_{e:\mathrm{in}(e)=\mathrm{out}(e)=i}S(\mathfrak{m}_e)\right\rangle\end{aligned}$$

has all poles contributed by $\mathcal{Y}_i(x+\varepsilon)$ cancelled by those contributed by $\frac{1}{\mathcal{Y}_i(x)}$. However, the $\mathcal{Y}$-functions coming from edges of the quiver still contribute poles so the desingularization process is not yet complete.

So far, all of our qq-characters had $\mathcal{Y}(x+\varepsilon)$ as their starting monomial (or $\mathcal{Y}_1(x+\varepsilon)$ or $\mathcal{Y}_2(x+\varepsilon)$), they were of degree one. In general, one may start with arbitrary monomials

$$\prod_{i\in\mathrm{Vert}_\gamma}\prod_{p=1}^{w_i}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p}), \tag{67}$$

where $\underline{w}\in\mathbb{Z}_{\geq 0}$ is the weight and $\sigma_{i,p}\in\mathbb{C}$ are shift constants. The expectation of (67) has poles, each pole coming from one of the $\mathcal{Y}$-functions in the product. By fixing a pair (I,P), $I\in\mathrm{Vert}_\gamma$, $P=1,\dots,w_I$, we single out one factor, $\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})$, in the starting monomial, i.e.

$$\prod_{i\in\mathrm{Vert}_\gamma}\prod_{p=1}^{w_i}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})=\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})\prod_{(i,p)\neq(I,P)}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})$$

The marked factor $\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})$ contributes poles to the expectation of (67). In particular, the pole $x_+=\phi_{I,\alpha,\square}-\sigma_{I,P}$ of

$$\mu_{\underline{\boldsymbol{\lambda}}_+}\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})[\underline{\boldsymbol{\lambda}}_+]\prod_{(i,p)\neq(I,P)}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})[\underline{\boldsymbol{\lambda}}_+],$$

where $\underline{\boldsymbol{\lambda}}_+=\underline{\boldsymbol{\lambda}}\cup(i,\alpha,\square)$, is cancelled by adding

$$\mu_{\underline{\boldsymbol{\lambda}}}\mathfrak{q}_IP_I(x+\sigma_{I,P})\frac{\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})[\underline{\boldsymbol{\lambda}}]}{A_I(x+\varepsilon+\sigma_{I,P})[\underline{\boldsymbol{\lambda}}]}\prod_{(i,p)\neq(I,P)}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})[\underline{\boldsymbol{\lambda}}_+]. \tag{68}$$

Now, the $\underline{\boldsymbol{\lambda}}_+$ in the (68) is undesirable because we want to sum over the partitions to get the expectation. We use

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}_+]=S(-x+\phi_{I,\alpha,\square})^{\delta_{i,I}}\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}],$$

which can be seen from (54). We also use $S(y-\varepsilon)=S(-y)$. Then, substituting $x=x_+$, (68) becomes

$$\mu_{\underline{\boldsymbol{\lambda}}}\mathfrak{q}_I P_I(x_+ + \sigma_{I,P})\frac{\mathcal{Y}_I(x_+ + \varepsilon + \sigma_{I,P})[\underline{\boldsymbol{\lambda}}]}{A_I(x_+ + \varepsilon + \sigma_{I,P})[\underline{\boldsymbol{\lambda}}]}\prod_{(i,p)\neq(I,P)}\left(S(\sigma_{i,p}-\sigma_{I,P})^{\delta_{i,I}}\mathcal{Y}_i(x_+ + \varepsilon + \sigma_{i,p})[\underline{\boldsymbol{\lambda}}]\right).$$

Thus, adding

$$\mathfrak{q}_I P_I(x+\sigma_{I,P})\frac{\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})}{A_I(x+\varepsilon+\sigma_{I,P})}\prod_{(i,p)\neq(I,P)}\left(S(\sigma_{i,p}-\sigma_{I,P})^{\delta_{i,I}}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})\right) \tag{69}$$

to the highest weight monomial (67) removes all poles contributed by $\mathcal{Y}_I(x+\varepsilon+\sigma_{I,P})$ to the expectation of (67). Of course, (67) also contains other $\mathcal{Y}$-functions (with other shifts $\sigma_{i,p}$) contributing poles to its expectation. These must be desingularized the same way. Furthermore, the term added in the desingularization, (69), contributes poles through the new $\mathcal{Y}$-functions introduced (in A and in $\prod_p$) which also need to be cancelled: If we want to cancel the poles contributed by $\mathcal{Y}_i(x)$ in

$$\left\langle \mathcal{Y}_i(x)\prod_{j\in\mathrm{Vert}_\gamma}\frac{\prod_{k=1}^{a_j}\mathcal{Y}_i(x_{j,k}^+)}{\prod_{k=1}^{b_j}\mathcal{Y}_j(x_{j,k}^-)}\right\rangle, \tag{70}$$

where $x_{i,k}^{\pm}=x+\mathfrak{k}_{i,k}^{\pm}$ for some function $\mathfrak{k}_{i,k}^{\pm}:\mathfrak{t}\to\mathbb{C}$ (for example something like $\mathfrak{k}_{i,k}^{\pm}=\varepsilon-\mathfrak{m}_e+\sigma_p$), we add

$$\mathfrak{q}_i P_i(x)\frac{\prod_{k=1}^{a_i}S(x_{i,k}^+-x)}{\prod_{k=1}^{b_i}S(x_{i,k}^--x)}\frac{\mathcal{Y}_i(x)}{A_i(x)}\prod_{j\in\mathrm{Vert}_\gamma}\frac{\prod_{k=1}^{a_j}\mathcal{Y}_i(x_{j,k}^+)}{\prod_{k=1}^{b_j}\mathcal{Y}_j(x_{j,k}^-)}. \tag{71}$$

Compared with (66), here the iWeyl reflection of $\mathcal{Y}_i$ is modified by the presence of the other $\mathcal{Y}$-functions in (70): Namely, we must also multiply by the S-functions.

At every step of this iterative process starting with the highest weight monomial (67), we obtain Laurent monomials in the shifted $\mathcal{Y}_i$-functions (treating the $P_i(x)$ and S-functions as coefficients). All $\mathcal{Y}_i$-functions in the numerator of a monomial must be desingularized by adding (71). This almost defines an algorithm for computing a qq-character starting from the highest weight monomial. What's missing are rules for how to treat multiplicities: The same Laurent monomial may be generated multiple times and it is not directly clear what its multiplicity should be in the qq-character. We will see this for the weight two qq-character of the A_1 quiver in Example 4.6. This algorithm can probably be completed by adapting the Frenkel–Mukhin (FM) algorithm [FM01], introduced for computing q-characters (which we will introduce in Section 4.5) from their highest weight monomials. The FM algorithm for q-characters is known to fail in certain cases, however to our knowledge the only example of failure is for a non-simply-laced simple Lie algebra: This does not correspond to any quiver and thus the algorithm remains a good candidate for qq-characters. In [FJM22], a notion of qq-character was defined entirely by way of FM algorithm, however this is in a purely representation theoretic context and it isn't proved that these qq-characters have polynomial expectation in quiver gauge theory.

Example 4.6. We demonstrate the desingularization process for a higher weight qq-character of quiver gauge theory $(A_1, n, 0)$ (as $m=0$, the matter polynomial P is equal to 1). We start off with the highest weight monomial

$$\mathcal{Y}(x+\sigma_1+\varepsilon)\mathcal{Y}(x+\sigma_2+\varepsilon).$$

$$
\begin{tikzcd}
& \mathcal{Y}(x+\sigma_1+\varepsilon)\mathcal{Y}(x+\sigma_2+\varepsilon) \arrow[dl] \arrow[dr] & \\
\mathfrak{q}S(\sigma_2-\sigma_1)\frac{\mathcal{Y}(x+\sigma_2+\varepsilon)}{\mathcal{Y}(x+\sigma_1)} \arrow[dr] & & \mathfrak{q}S(\sigma_1-\sigma_2)\frac{\mathcal{Y}(x+\sigma_1+\varepsilon)}{\mathcal{Y}(x+\sigma_2)} \arrow[dl] \\
& \mathfrak{q}^2\frac{1}{\mathcal{Y}(x+\sigma_1)\mathcal{Y}(x+\sigma_2)} &
\end{tikzcd}
$$

Figure 1: Hasse diagram of the qq-character $\mathcal{X}_{2,(\sigma_1,\sigma_2)}(x)$ of quiver gauge theory (A_1, n, m)

We cancel the poles by adding

$$\mathfrak{q}S(\sigma_2-\sigma_1)\frac{\mathcal{Y}(x+\sigma_1+\varepsilon)}{A(x+\sigma_1+\varepsilon)}\mathcal{Y}(x+\sigma_2+\varepsilon) = \mathfrak{q}S(\sigma_2-\sigma_1)\frac{\mathcal{Y}(x+\sigma_2+\varepsilon)}{\mathcal{Y}(x+\sigma_1)}, \tag{72}$$

and

$$\mathfrak{q}S(\sigma_1-\sigma_2)\frac{\mathcal{Y}(x+\sigma_2+\varepsilon)}{A(x+\sigma_2+\varepsilon)}\mathcal{Y}(x+\sigma_1+\varepsilon) = \mathfrak{q}S(\sigma_1-\sigma_2)\frac{\mathcal{Y}(x+\sigma_1+\varepsilon)}{\mathcal{Y}(x+\sigma_2)}. \tag{73}$$

Now, the numerator $\mathcal{Y}$-functions of (72) and (73) contribute poles of their own to the expectation. These need to be cancelled too. We add

$$\mathfrak{q}^2\frac{S(\sigma_2-\sigma_1)}{S(\sigma_1-\sigma_2-\varepsilon)}\frac{1}{\mathcal{Y}(x+\sigma_1)\mathcal{Y}(x+\sigma_2)}.$$

(equivalently with $S(\sigma_1-\sigma_2)/S(\sigma_2-\sigma_1-\varepsilon)$) and remark that the S-functions cancel by the property $S(y-\varepsilon) = S(-y)$. Then, the full qq-character of $(A_1, n, 0)$, with shifts (σ_1, σ_2),

$$\begin{aligned}\mathcal{X}_{2,(\sigma_1,\sigma_2)}(x) = \mathcal{Y}(x+\sigma_1+\varepsilon)\mathcal{Y}(x+\sigma_2+\varepsilon) + \mathfrak{q}S(\sigma_2-\sigma_1)\frac{\mathcal{Y}(x+\sigma_2+\varepsilon)}{\mathcal{Y}(x+\sigma_1)} \\ + \mathfrak{q}S(\sigma_1-\sigma_2)\frac{\mathcal{Y}(x+\sigma_1+\varepsilon)}{\mathcal{Y}(x+\sigma_2)} + \mathfrak{q}^2\frac{1}{\mathcal{Y}(x+\sigma_1)\mathcal{Y}(x+\sigma_2)},\end{aligned} \tag{74}$$

has as its expectation a polynomial of degree two. This process of recursive desingularization is summarized in a Hasse diagram (fig. 1). The reason that the sum of all terms in the diagram has non-singular expectation is that for every term

- for each $\mathcal{Y}$-function in its numerator, there is another term desingularizing it, connected by an arrow pointing away from the term,
- for each $\mathcal{Y}$-function in its denominator, there is another term desingularizing it, connected by an arrow pointing toward the term.

More concretely, observe that the terms of type $\frac{\mathcal{Y}}{\mathcal{Y}}$ have both an incoming and an outgoing arrow, while the term of type $\mathcal{Y}\mathcal{Y}$ has two outgoing arrows and the term of type $\frac{1}{\mathcal{Y}\mathcal{Y}}$ has two incoming arrows.

Recall that $S(x) = \frac{(x+\varepsilon_1)(x+\varepsilon_2)}{x(x+\varepsilon)}$. Thus, (74) appears to have singularities at $\sigma_1 = \sigma_2$. These only arise due to the rational function (74) not being completely reduced. Indeed, the limit

$$\mathcal{X}_{2,(0,0)}(x) := \lim_{(\sigma_1,\sigma_2)\to(0,0)} \mathcal{X}_{2,(\sigma_1,\sigma_2)}(x)$$

is finite. It involves derivatives of the $\mathcal{Y}$-functions. Nekrasov [Nek16, equation 150] gives it as

$$\mathcal{Y}(x+\varepsilon)^2\left(1-\mathfrak{q}\frac{\varepsilon_1\varepsilon_2}{\varepsilon}\frac{d}{dx}\left(\frac{1}{\mathcal{Y}(x)\mathcal{Y}(x+\varepsilon)}\right)\right)+2\mathfrak{q}\frac{\mathcal{Y}(x+\varepsilon)}{\mathcal{Y}(x)}\left(1-\frac{\varepsilon_1\varepsilon_2}{\varepsilon^2}\right)+\mathfrak{q}^2\frac{1}{\mathcal{Y}(x)^2}.$$

Note that in the limit $\varepsilon_2 = 0$, this no longer involves derivatives. This is a general principle for qq-characters, as we will prove in Lemma 4.13.

For general quivers the iterative desingularization may never terminate, as we now demonstrate for the $\hat{A}_0$ quiver:

Example 4.7. As there is only one vertex, there is also just one $\mathcal{Y}$-observable. But the edge contributes an edge mass $\mathfrak{m}$ and edge $\mathcal{Y}$-observables in the desingularization:

$$\frac{\mu_{\boldsymbol{\lambda}_+}}{\mu_{\boldsymbol{\lambda}}} = (-1)^\kappa \mathfrak{q} S(\mathfrak{m}) \lim_{x\to x_+}\left(\frac{P(x)}{\mathcal{Y}(x)[\boldsymbol{\lambda}]\mathcal{Y}(x+\varepsilon)[\boldsymbol{\lambda}_+]}\mathcal{Y}(x+\mathfrak{m}+\varepsilon)\mathcal{Y}(x-\mathfrak{m})\right)$$

where $\boldsymbol{\lambda}_+ = \boldsymbol{\lambda}\cup(\alpha_+,\square_+)$ and $x_+ = \phi_{\alpha_+,\square_+}$. We start with $\langle\mathcal{Y}(x+\varepsilon)\rangle$. The first desingularization produces

$$\left\langle \mathcal{Y}(x+\varepsilon)+\mathfrak{q}S(\mathfrak{m})\frac{P(x)}{\mathcal{Y}(x)}\mathcal{Y}(x+\mathfrak{m}+\varepsilon)\mathcal{Y}(x-\mathfrak{m})\right\rangle$$

The edge contributes two $\mathcal{Y}$-functions in the numerator of the $\mathfrak{q}$-term. These need to be desingularized as well. The second desingularization is

$$\begin{aligned}\frac{\mathcal{Y}(x+\mathfrak{m}+\varepsilon)\mathcal{Y}(x-\mathfrak{m})}{\mathcal{Y}(x)} &\to \mathfrak{q}S(\mathfrak{m})\left(\frac{\mathcal{Y}(x-\mathfrak{m})}{\mathcal{Y}(x)}\frac{\mathcal{Y}(x+2\mathfrak{m}+\varepsilon)\mathcal{Y}(x)}{\mathcal{Y}(x+\mathfrak{m})}+\frac{\mathcal{Y}(x+\mathfrak{m}+\varepsilon)}{\mathcal{Y}(x)}\frac{\mathcal{Y}(x)\mathcal{Y}(x-2\mathfrak{m}-\varepsilon)}{\mathcal{Y}(x-\mathfrak{m}-\varepsilon)}\right)\\ &= \mathfrak{q}S(\mathfrak{m})\left(\frac{\mathcal{Y}(x-\mathfrak{m})\mathcal{Y}(x+2\mathfrak{m}+\varepsilon)}{\mathcal{Y}(x+\mathfrak{m})}+\frac{\mathcal{Y}(x+\mathfrak{m}+\varepsilon)\mathcal{Y}(x-2\mathfrak{m}-\varepsilon)}{\mathcal{Y}(x-\mathfrak{m}-\varepsilon)}\right)\end{aligned}$$

(as well as some additional factors of S-functions as in (71).) We see that both new terms contain two $\mathcal{Y}$-functions in the numerator, which will need to be desingularized again. This carries on forever: Every time a $\mathcal{Y}$-function in the numerator is desingularized by the prescription

$$\mathcal{Y}(x+\varepsilon)\to\frac{\mathcal{Y}(x+\mathfrak{m}+\varepsilon)\mathcal{Y}(x-\mathfrak{m})}{\mathcal{Y}(x)},$$

in its stead two new $\mathcal{Y}$-functions are introduced in the numerator, while only one additional $\mathcal{Y}$-function from the numerator is cancelled by the $\mathcal{Y}$ introduced in the denominator. Thus, there will always be two $\mathcal{Y}$-functions in the numerator, and the process never terminates for the $\hat{A}_0$ quiver (but one may generate a series by continuing). We will compute this quiver's fundamental qq-character using the definition as an integral over the quiver variety in Section 4.4.

A note on different conventions. When engaging with the literature on qq-characters, one should be aware that not everyone uses the same conventions as Nekrasov [Nek16] and we do. In particular, Kimura [Kim21] uses another convention in his book and articles. The origin of this lies in a different definition of the actions (19), (20) and the bundle $\mathbf{B}$ (cf. (42)) over the instanton moduli space. To get from one convention to the other, one need only reverse all edges and substitute the edge masses $\mathfrak{m}_e \to -\mathfrak{m}_e$. Kimura's convention appears to play nicer with the q-characters of Section 4.5.

Non-perturbative Dyson–Schwinger equations and qq-characters in K-theory. The K-theory partition function $\mathcal{Z}^K_{\text{inst}}$ (cf. (46)) defines another complex measure on $\mathfrak{P}(\underline{n})$. In K-theory, the Chern polynomial is replaced by

$$c^T_z(\mathbf{E}) \coloneqq \wedge_{-z^{-1}}\mathbf{E}^* \coloneqq \sum_{k=1}^{\operatorname{rk}\mathbf{E}}(-z^{-1}) \wedge^k \mathbf{E}^* \tag{75}$$

(an element of the complexified K-theory) which reduces to the K-theory Euler class $\epsilon^T(\mathbf{E})$ in the limit $z = 1$. Identifying $c^T_z(\mathbf{E})$ with its image under Ch^T, it is expressed in terms of the formal equivariant Chern roots as

$$c^T_z(\mathbf{E}) = \prod_{k=1}^{\operatorname{rk}\mathbf{E}}(1 - z^{-1}e^{-r_i}).$$

Using this, we define the K-theory versions of the functions $S, P_i, \mathcal{Y}_i$:

$$S(z) = \frac{1}{c^T_z(Q)} = \frac{(1-q_1^{-1}z^{-1})(1-q_2^{-1}z^{-1})}{(1-z^{-1})(1-q^{-1}z^{-1})}, \qquad P_i(z) = c^T_z(M_i) = \prod_{f=1}^{m_i}(1 - e^{-\mathfrak{m}_{i,f}}z^{-1})$$

$$\mathcal{Y}_i(z)[\underline{\boldsymbol{\lambda}}] = c^T_z(S_i[\underline{\boldsymbol{\lambda}}]^*) = \prod_{\alpha=1}^{n_i}\frac{\prod_{\square\in\partial_+\lambda^{(i,\alpha)}}(1 - e^{\mathfrak{a}_{i,\alpha}}q_1^{s_1-1}q_2^{s_2-1}z^{-1})}{\prod_{\square\in\partial_-\lambda^{(i,\alpha)}}(1 - e^{\mathfrak{a}_{i,\alpha}}q_1^{s_1}q_2^{s_2}z^{-1})},$$

where $q_{1,2} = e^{\varepsilon_{1,2}}$, $q = q_1q_2 = e^{\varepsilon}$. The non-perturbative transformation of the measure under $\underline{\boldsymbol{\lambda}} \to \underline{\boldsymbol{\lambda}} \cup (i_+, \alpha_+, \square_+)$, where $z_+ = e^{x_+} = e^{\phi_{i_+,\alpha_+,\square_+}}$, is then

$$\frac{\mu_{\underline{\boldsymbol{\lambda}}_+}}{\mu_{\underline{\boldsymbol{\lambda}}}} = (-1)^{\kappa_{i_+}}z_+^{\kappa_{i_+}}f_{i_+}\mathfrak{q}_{i_+}P_{i_+}(z_+)\frac{\prod_{e:s\to i_+}\mathcal{Y}_s(qe^{\mathfrak{m}_e}z_+)\prod_{e:i_+\to t}\mathcal{Y}_t(e^{-\mathfrak{m}_e}z_+)}{\mathcal{Y}_{i_+}(z_+)[\underline{\boldsymbol{\lambda}}]\mathcal{Y}_{i_+}(qz_+)[\underline{\boldsymbol{\lambda}}_+]}\prod_{e:i_+\to i_+}S(e^{\mathfrak{m}_e}). \tag{76}$$

where $\kappa_{i_+} = -n_{i_+} + \sum_{e:i_+\to t}n_t$ and $f_{i_+} = e^{\sum_{\alpha=1}^{n_{i_+}}\mathfrak{a}_{i_+,\alpha} - \sum_{e:i_+\to t}\sum_{\alpha=1}^{n_t}\mathfrak{a}_{t,\alpha}}$. Note that the factor $z_+^{\kappa_{i_+}}f_{i_+}$ was not present in the non-perturbative transformation of the cohomology measure (52); it stems from the fact that the K-theory Euler class of a representation's dual is $\epsilon^T(V^*) = \prod_{w\in\mathrm{Weights}(V)}(-e^w)\epsilon^T(V)$ whereas in cohomology it is $\epsilon^T(V^*) = (-1)^{\dim V}\epsilon^T(V)$. Equation (76) gives rise to non-perturbative Dyson–Schwinger equations and K-theory qq-characters. For example, the first fundamental qq-character for the $(A_2, \underline{n}, \underline{m})$ quiver gauge theory is

$$\mathcal{X}^K_{\underline{\delta}_1,0}(z) = \mathcal{Y}_1(qz) + \mathfrak{q}_1z^{n_2-n_1}f_1\frac{P_1(z)}{\mathcal{Y}_1(z)}\mathcal{Y}_2(e^{-\mathfrak{m}_e}z) + \mathfrak{q}_1\mathfrak{q}_2z^{-n_1}f_1f_2\frac{P_1(z)P_2(q^{-1}e^{-\mathfrak{m}_e}z)}{\mathcal{Y}_2(q^{-1}e^{-\mathfrak{m}_e}z)}.$$

Like its cohomology analogue (64), its expectation (under the K-theory measure) is non-singular. Unlike for cohomology, it is not polynomial in z but rather polynomial in z^{-1} (a Laurent polynomial with no positive degree powers) of degree n_1.

In general, one obtains the K-theory qq-character from the cohomology qq-character by replacing $\mathfrak{q}_i \to \mathfrak{q}_iz^{\kappa_i}f_i$ and exponentiating the arguments of the various functions (i.e. $P_i(x + \mathfrak{m}_e - \varepsilon) \to P_i(q^{-1}e^{\mathfrak{m}_e}z)$ etc.)

4.3 The Dyson–Schwinger equation in the $\hbar \to 0$ limit

Our goal now is to study the limit of non-perturbative Dyson–Schwinger equations for $\varepsilon_1, \varepsilon_2 \to 0$. In this limit, the Dyson–Schwinger equation defines a spectral curve similar to those of random matrix theory (e.g. [BEO13]) encoding the asymptotic expansions in N^2, where N is the size of the matrix, of the free energy $-\log Z$ and the correlation functions. Similarly, Nekrasov [Nek03] computed the prepotential of an $\mathcal{N}=2$ gauge theory, defined in terms of the geometry of the Seiberg–Witten curve [SW94], as the limit $\lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2\log\mathcal{Z}_{\text{inst}}$.

We work with the (A_1,n,m) quiver gauge theory with β function $\beta\le 0$ (here $\beta=m-2n$). For the sake of simplicity, we restrict the parameters $\varepsilon_1,\varepsilon_2$ of the measure to $-\varepsilon_1=\varepsilon_2=\hbar$. Regarding the other parameters $\underline{\mathfrak{a}},\underline{\mathfrak{m}}$ as fixed, the measure $\mathbb{P}_\hbar$ on the set of partitions $\mathfrak{P}(n)$ depends only on $\hbar$. The limit $\varepsilon_1,\varepsilon_2\to 0$ is then investigated by letting $\hbar\to 0$. In Proposition 4.3, we gave the theory's fundamental Dyson–Schwinger equation

$$\left\langle \mathcal{Y}(x)+\mathfrak{q}\frac{P(x)}{\mathcal{Y}(x)}\right\rangle_\hbar = T(x),$$

where T is an $\hbar$-dependent deterministic polynomial of degree n.

Both the measure $\mathbb{P}_\hbar$ and the rational functions $\mathcal{Y}(x)[\boldsymbol{\lambda}]$ depend on $\hbar$, and

$$\lim_{\hbar\to 0}\langle\mathcal{Y}(x)\rangle_\hbar \neq \lim_{\hbar\to 0}\left\langle \lim_{\hbar\to 0}\mathcal{Y}(x)\right\rangle_\hbar.$$

While $\mathbb{P}_\hbar$ certainly doesn't converge on the space of partitions, it converges on a larger space that we embed the partitions in, e.g. the space of profiles of partitions as in [NO06].

Let us assume that $\mathcal{Y}(x),T(x)$ have limits

$$\overline{\mathcal{Y}}(x)=\lim_{\hbar\to 0}\langle\mathcal{Y}(x)\rangle_\hbar,\qquad \overline{T}(x)=\lim_{\hbar\to 0}\langle T(x)\rangle_\hbar.$$

Then the Dyson–Schwinger equation has a small $\hbar$ limit:

$$\overline{\mathcal{Y}}(x)-\overline{T}(x)+\mathfrak{q}\frac{P(x)}{\overline{\mathcal{Y}}(x)}=0$$

or equivalently,

$$\overline{\mathcal{Y}}(x)^2-\overline{T}(x)\overline{\mathcal{Y}}(x)+\mathfrak{q}P(x)=0. \tag{77}$$

4.3.1 Random partitions as random matrices

We start by giving a different expression for the $\mathcal{Y}_i$-observables.

Lemma 4.8. *When $-\varepsilon_1=\varepsilon_2=\hbar$,*

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}]=\prod_{\alpha=1}^{n_i}\left[(x-\mathfrak{a}_{i\alpha}+\hbar L^{i\alpha})\prod_{s=1}^{L^{i\alpha}}\frac{x-\mathfrak{a}_{i\alpha}-\hbar(h_s^{i\alpha}+1)}{x-\mathfrak{a}_{i\alpha}-\hbar h_s^{i\alpha}}\right],$$

where $L^{i\alpha}:=\mathrm{length}(\gamma^{i\alpha})$ and $h_s^{i\alpha}:=\lambda_s^{i\alpha}-s$.

Proof. We start with $\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}]$ as in (54) (writing $c_\square=(s_1-1)\varepsilon_1+(s_2-1)\varepsilon_2=\phi_{i\alpha\square}-\mathfrak{a}_{i\alpha}$):

$$\mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}]=\prod_{\alpha=1}^{n_i}\left[(x-\mathfrak{a}_{i\alpha})\prod_{\square\in\lambda^{i\alpha}}\frac{x-\mathfrak{a}_{i\alpha}-c_\square-\varepsilon_1}{x-\mathfrak{a}_{i\alpha}-c_\square-\varepsilon}\frac{x-\mathfrak{a}_{i\alpha}-c_\square-\varepsilon_2}{x-\mathfrak{a}_{i\alpha}-c_\square}\right] \tag{78}$$

$$= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha}) \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - c_{(s,0)} - \varepsilon}{x - \mathfrak{a}_{i\alpha} - c_{(s,\lambda_s^{i\alpha})} - \varepsilon} \frac{x - \mathfrak{a}_{i\alpha} - c_{(s,\lambda_s+1)}}{x - \mathfrak{a}_{i\alpha} - c_{(s,1)}} \right] \tag{79}$$

$$= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha}) \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - s\varepsilon_1}{x - \mathfrak{a}_{i\alpha} - (s-1)\varepsilon_1} \frac{x - \mathfrak{a}_{i\alpha} - (s-1)\varepsilon_1 - \lambda_s \varepsilon_2}{x - \mathfrak{a}_{i\alpha} - (s-1)\varepsilon_1} \right] \tag{80}$$

$$= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha}) \frac{x - \mathfrak{a}_{i\alpha} - L^{i\alpha}\varepsilon_1}{x - \mathfrak{a}_{i\alpha}} \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - (s-1)\varepsilon_1 - \lambda_s \varepsilon_2}{x - \mathfrak{a}_{i\alpha} - (s-1)\varepsilon_1} \right], \tag{81}$$

where, to get from (78) to (79) and from (80) to (81), we performed telescopic products. As $\varepsilon_2 = -\varepsilon_1 = \hbar$,

$$\begin{aligned} \mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}] &= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha}) \frac{x - \mathfrak{a}_{i\alpha} + \hbar L^{i\alpha}}{x - \mathfrak{a}_{i\alpha}} \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - \hbar(h_s^{i\alpha} + 1)}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}} \right] \\ &= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha} + \hbar L^{i\alpha}) \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - \hbar(h_s^{i\alpha} + 1)}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}} \right]. \end{aligned}$$

□

Using this, we now build a bridge to random matrix theory. Our random object is a random $\underline{n}$-colored partition $\underline{\boldsymbol{\lambda}}$. We convert this into multiple random matrices M_i, one for each vertex i, with diagonal entries $\mathfrak{a}_{i\alpha} + \hbar h_s^{i\alpha}$ where $\alpha = 1, \ldots, n_i$ and $s = 1, \ldots, L^{i\alpha}$. This matrix's resolvent (or rather its rescaled trace $\hbar \operatorname{tr} \frac{1}{x - M_i}$) is

$$W_i(x) := \hbar \sum_{\alpha=1}^{n_i} \sum_{s=1}^{L^{i\alpha}} \frac{1}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}}.$$

The reason for rescaling by $\hbar$ is that a typical size of random partition under $\mathbb{P}_\hbar$ is $k \sim \frac{1}{\hbar^2}$ and the typical length is $L^\alpha \sim \frac{1}{\hbar}$. We seek to transform the Dyson–Schwinger equation for $\mathcal{Y}_i(x)$ into an equation for $W_i(x)$.

Lemma 4.9. *In terms of the resolvent of the random matrix M_i, the $\mathcal{Y}_i$-observables are*

$$\mathcal{Y}_i(x) = Q_i(x) \exp\left(\sum_{r \geq 0} \frac{(-1)^{r+1} \hbar^r}{(m+1)!} W_i^{(r)}(x) \right),$$

where $Q(x) := \prod_\alpha (x - \mathfrak{a}_\alpha + \hbar L^\alpha)$ is a random, $\hbar$-dependent polynomial of degree n.

Proof. From Lemma 4.8, we know

$$\begin{aligned} \mathcal{Y}_i(x)[\underline{\boldsymbol{\lambda}}] &= \prod_{\alpha=1}^{n_i} \left[(x - \mathfrak{a}_{i\alpha} + \hbar L^{i\alpha}) \prod_{s=1}^{L^{i\alpha}} \frac{x - \mathfrak{a}_{i\alpha} - \hbar(h_s^{i\alpha} + 1)}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}} \right] \\ &= \exp\left[\sum_{\alpha=1}^{n_i} \log(x - \mathfrak{a}_{i\alpha} + \hbar L^{i\alpha}) + \sum_{\alpha=1}^{n_i} \sum_{s=1}^{L^{i\alpha}} \log\left(1 - \frac{\hbar}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}} \right) \right]. \end{aligned} \tag{82}$$

Using the series expansion $\log(1-\Delta) = -\sum_{m=1}^{\infty} \frac{1}{m}\Delta^m$, and the derivatives

$$\frac{d^r W_i(x)}{dx^r} = \hbar \sum_{\alpha=1}^{n} \sum_{s=1}^{L^{i\alpha}} \frac{(-1)^r r!}{(x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha})^{r+1}}$$

of the resolvent,

$$\begin{aligned}
\sum_{\alpha,s} \log\left(1 - \frac{\hbar}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}}\right) &= -\sum_{r\geq 1}\sum_{\alpha,s} \frac{1}{r}\left(\frac{\hbar}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}}\right)^r \\
&= -\sum_{r\geq 0}\sum_{\alpha,s} \frac{1}{r+1}\left(\frac{\hbar}{x - \mathfrak{a}_{i\alpha} - \hbar h_s^{i\alpha}}\right)^{r+1} \\
&= \sum_{r\geq 0} \frac{(-1)^{r+1}\hbar^r}{(r+1)!} W_i^{(r)}(x). \qquad (83)
\end{aligned}$$

The lemma is then just a combination of (82) and (83). □

4.3.2 The $\hbar \to 0$ limit

The $\hbar \to 0$ limit of the Nekrasov partition function was first studied by Nekrasov and Okounkov in [NO06]. They identified partitions with their profiles, certain functions $\mathbb{R} \to \mathbb{R}$, and argued that the distribution of the $\hbar h_s^{ia}$ converges in an appropriate sense as $\hbar \to 0$. The Dyson–Schwinger equation may serve as an alternative starting point for such analyses.

In (77), we gave the small $\hbar$ limit of the Dyson–Schwinger equation as an equation for

$$\overline{\mathcal{Y}}(x) = \lim_{\hbar\to 0} \langle \mathcal{Y}(x)\rangle_\hbar = \lim_{\hbar\to 0} \left\langle Q(x)\exp(W(x) - \hbar W'(x)/2 \pm \dots)\right\rangle_\hbar.$$

For some $\hbar$-dependent observables $A_\hbar, B_\hbar$, we assume that

$$\lim_{\hbar\to 0} \langle A_\hbar B_\hbar\rangle_\hbar = \lim_{\hbar\to 0} \langle A_\hbar\rangle_\hbar \lim_{\hbar\to 0} \langle B_\hbar\rangle_\hbar\,,$$

(and analogously for higher products) which is justified by the concentration of the measure at the saddle point in $\hbar \to 0$ limit. Such properties are common in random matrix theory [BGG17; AGZ10]. Using this self-averaging property,

$$\begin{aligned}
\overline{\mathcal{Y}}(x) &= \overline{Q}(x)\exp\left(\lim_{\hbar\to 0}\langle W(x)\rangle_\hbar - \lim_{\hbar\to 0}\hbar\,\langle W'(x)/2\rangle_\hbar \pm \dots\right) \\
&= \overline{Q}(x)e^{\overline{W}(x)}
\end{aligned}$$

where

$$\overline{Q}(x) \coloneqq \lim_{\hbar\to 0}\langle Q(x)\rangle_\hbar\,, \qquad \overline{W}(x) \coloneqq \lim_{\hbar\to 0}\langle W(x)\rangle_\hbar\,.$$

Note that $\overline{Q}(x)$ is a polynomial of degree n. Expressed in terms of the resolvent $\overline{W}(x)$, the $\hbar \to 0$ limit (77) of the Dyson–Schwinger equation becomes

$$\overline{Q}(x)^2 e^{2\overline{W}(x)} - \overline{T}(x)\overline{Q}(x)e^{\overline{W}(x)} + \mathfrak{q}P(x) = 0.$$

The spectral curve. Substituting $\mathcal{Y}(x) \to y$ in (77) yields an algebraic equation

$$F(x, y) := y^2 - \overline{T}(x)y + \mathfrak{q}P(x) = 0$$

defining an algebraic curve, called the spectral curve. For generic coefficients of the polynomials $\overline{T}(x)$ (degree n) and $P(x)$ (degree m), the curve's genus may be computed using the Newton polygon of the polynomial $F(x, y)$ using the standard method outlined in [Kho92]. Clearly, in this case the Newton polygon contains $n - 1$ interior points, thus that is the spectral curve's genus.

Because the spectral curve contains as part of its geometric structure the exponential of the $\hbar \to 0$ resolvent, $e^{\overline{W}(x)} = \frac{\overline{\mathcal{Y}}(x)}{\overline{Q}(x)}$, we expect that like in random matrix theory it encodes much information about the distributions of the random partitions in the limit $\hbar \to 0$, via the eigenvalues $\mathfrak{a}_\alpha + \hbar h_s^\alpha$ of the random matrix M.

Another example. The fundamental qq-characters of other quiver gauge theories also define spectral curves. An $\underline{n}$-colored partition consists of an n_i-colored partition for each vertex i, each identified with a random diagonal matrix. Thus, a quiver gauge theory defines a system of multiple random matrices, one for each vertex. Each of these has its own resolvent function $W_i(x)$ and its own fundamental Dyson–Schwinger equation.

For example, for $(A_2, \underline{n}, \underline{m})$ with β function $\beta_1, \beta_2 \le 0$, we have the Dyson–Schwinger equations (setting $\varepsilon = \mathfrak{m}_e = 0$)

$$\begin{aligned}
\left\langle \mathcal{Y}_1(x) + \mathfrak{q}_1 \frac{P_1(x)}{\mathcal{Y}_1(x)} \mathcal{Y}_2(x) + \mathfrak{q}_1\mathfrak{q}_2 \frac{P_1(x)P_2(x)}{\mathcal{Y}_2(x)} \right\rangle &= T_1(x), \\
\left\langle \mathcal{Y}_2(x) + \mathfrak{q}_2 \frac{P_2(x)}{\mathcal{Y}_2(x)} \mathcal{Y}_1(x) + \mathfrak{q}_1\mathfrak{q}_2 \frac{P_1(x)P_2(x)}{\mathcal{Y}_1(x)} \right\rangle &= T_2(x)
\end{aligned}$$

($\deg T_i = n_i$, $\deg P_i = m_i$) whose $\hbar \to 0$ limits are (setting $y_1 = \overline{\mathcal{Y}}_1(x)$, $y_2 = \overline{\mathcal{Y}}_2(x)$)

$$\begin{aligned}
y_1^2 y_2 + \mathfrak{q}_1 P_1(x) y_2^2 + \mathfrak{q}_1\mathfrak{q}_2 P_1(x)P_2(x) y_1 - \overline{T}_1(x) y_1 y_2 &= 0, \\
y_2^2 y_1 + \mathfrak{q}_2 P_2(x) y_1^2 + \mathfrak{q}_2\mathfrak{q}_1 P_2(x)P_1(x) y_2 - \overline{T}_2(x) y_2 y_1 &= 0,
\end{aligned}$$

the solutions (x, y_1, y_2) again defining a spectral curve, with $e^{\overline{W}_1(x)} = \frac{\mathcal{Y}_1(x)}{\overline{Q}_1(x)}$ and $e^{\overline{W}_2(x)} = \frac{\mathcal{Y}_2(x)}{\overline{Q}_2(x)}$ being part of the geometric structure.

4.4 The geometric definition of qq-characters

Nekrasov [Nek16, Sec. 8.3, 8.4] gave a formula for the qq-characters of an arbitrary quiver gauge theory $(\gamma, \underline{n}, \underline{m})$, as generating functions of equivariant integrals over the quiver varieties $\mathfrak{Q}(\underline{w}, \underline{v})$.

Given any quiver γ and two dimension vectors $\underline{w}, \underline{v} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$, we define the vector spaces $W_i = \mathbb{C}^{w_i}$, $V_i = \mathbb{C}^{v_i}$ and

$$X_\gamma := \bigoplus_{e \in \mathrm{Edges}_\gamma} \mathrm{Hom}(V_{\mathrm{in}(e)}, V_{\mathrm{out}(e)}) \oplus \mathrm{Hom}(V_{\mathrm{out}(e)}, V_{\mathrm{in}(e)}) \oplus \bigoplus_{i \in \mathrm{Vert}_\gamma} \mathrm{Hom}(V_i, W_i) \oplus \mathrm{Hom}(W_i, V_i),$$

and denote elements by $(B_{e,+}, B_{e,-}, I_i, J_i)_{e\in\mathrm{Edges}_\gamma, i\in\mathrm{Vert}_\gamma}$. This space has an action by $U(\underline{v}) := \bigtimes_{i\in\mathrm{Vert}_\gamma} U(v_i)$,

$$(g_i)_{i\in\mathrm{Vert}_\gamma}: \qquad (B_{e,+}, B_{e,-}, I_i, J_i) \mapsto (g_{\mathrm{out}(e)}B_{e,+}g^{-1}_{\mathrm{in}(e)}, g_{\mathrm{in}(e)}B_{e,-}g^{-1}_{\mathrm{out}(e)}, g_iI_i, J_ig_i^{-1}).$$

Introducing a certain hyperkähler moment map $\mu = \frac{\mathrm{i}}{2}\mu^r + \mu^c$ on X_γ, the quiver variety is defined as the hyperkähler quotient

$$\mathfrak{Q}(\underline{w}, \underline{v}) = \mathfrak{Q}_{\gamma,\underline{\zeta}}(\underline{w}, \underline{v}) := \{x \in X_\gamma \mid \mu^r(x) = \underline{\zeta}, \mu^c(x) = 0\}/U(\underline{v}),$$

where the choice of $\underline{\zeta} \in \mathbb{R}_{>0}^{\mathrm{Vert}_\gamma}$ is unimportant. Quiver varieties were introduced by Nakajima [Nak94a], a comprehensive introduction is [Kir16].

We define $G_{\underline{w}} := \bigtimes_{i\in\mathrm{Vert}_\gamma} GL(w_i, \mathbb{C})$. It acts on the quiver variety through $(I_ig_i, g_i^{-1}J_i)$. Its complexified maximal torus is $T_{\underline{w}}$, equivariant parameters are called $\underline{\sigma} \in \bigoplus_{i\in\mathrm{Vert}_\gamma} \mathbb{C}^{w_i}$ (identified with the shift parameters $\sigma_{i,p}$). We also define actions

$$\begin{aligned}
\mathbb{C}^\times &\curvearrowright X_\gamma, & u\cdot(B_{e,+}, B_{e,-}, I_i, J_i) &= (uB_{e,+}, B_{e,-}, I_i, uJ_i)\\
(\mathbb{C}^\times)^{\mathrm{Edges}_\gamma} &\curvearrowright X_\gamma, & (u_e)_{e\in\mathrm{Edges}_\gamma}\cdot(B_{e,+}, B_{e,-}, I_i, J_i) &= (u_eB_{e,+}, u_e^{-1}B_{e,-}, I_i, J_i)\\
(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma} &\curvearrowright X_\gamma, & (u_i)_{i\in\mathrm{Vert}_\gamma}\cdot(B_{e,+}, B_{e,-}, I_i, J_i) &= (u_{\mathrm{out}(e)}B_{e,+}u^{-1}_{\mathrm{in}(e)}, u_{\mathrm{in}(e)}B_{e,-}u^{-1}_{\mathrm{out}(e)}, I_i, J_i)
\end{aligned}$$

Through this action, $(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma}$ can be regarded as a subgroup of $(\mathbb{C}^\times)^{\mathrm{Edges}_\gamma}$, and the quotient is isomorphic to the torus $(\mathbb{C}^\times)^{\mathrm{Edges}_\gamma}/(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma} \cong (\mathbb{C}^\times)^{b_1(\gamma)}$. So overall, we have an action of $G_{\underline{w}} \times (\mathbb{C}^\times)^{b_*(\gamma)}$ on $\mathfrak{Q}(\underline{w}, \underline{v})$ (where $b_*(\gamma) = b_0(\gamma) + b_1(\gamma)$ and, as we only consider connected quivers, $b_0(\gamma) = 1$).

Recall the group acting on the moduli space $\mathfrak{M}(\underline{n}, \underline{k})$ of the quiver gauge theory $(\gamma, \underline{n}, \underline{m})$:

$$\bigtimes_{i\in\mathrm{Vert}_\gamma} U(n_i) \times \left(\left(\bigtimes_{i\in\mathrm{Vert}_\gamma} U(m_i) \times U(1)^{\mathrm{Edges}_\gamma}\right) \Big/ U(1)^{\mathrm{Vert}_\gamma}\right) \times U(1)^2$$

Its complexified maximal torus is T. We define a homomorphism

$$T \to (\mathbb{C}^\times)^{b_*(\gamma)}, \quad (e^{\underline{\mathfrak{a}}}, e^{\underline{\mathfrak{m}}}, e^{\vec{\varepsilon}}) \mapsto \left(e^{-\varepsilon}, [(e^{\mathfrak{m}_e+\varepsilon})_{e\in\mathrm{Edges}_\gamma}]\right).$$

We then pull back the action of $(\mathbb{C}^\times)^{b_*(\gamma)}$ on $\mathfrak{Q}(\underline{w}, \underline{v})$ to T, thus turning $\mathfrak{Q}(\underline{w}, \underline{v})$ into a $T_{\underline{w}} \times T$-manifold. In (90) we will compare this action for the $\hat{A}_0$ case, where the quiver variety is the moduli space $\mathfrak{M}(n, k)$ with the action defined in Section 3.

Over the quiver variety $\mathfrak{Q}(\underline{w}, \underline{v})$ are the trivial bundles $\mathbf{W}_i$ and the bundles $\mathbf{V}_i$, $i \in \mathrm{Vert}_\gamma$, defined as the bundles associated to the fundamental representations $G_{\underline{v}} \to V_i$. They are $G_{\underline{w}} \times (\mathbb{C}^\times)^{b_*(\gamma)}$-equivariant. (Our quivers are all connected, so that $b_0(\gamma) = 1$ and $b_1(\gamma) = \#\mathrm{Edges}_\gamma - \#\mathrm{Vert}_\gamma + 1$.) We then define the virtual equivariant bundle

$$\mathbf{C}_i = \mathbf{W}_i \ominus \mathbf{V}_i \ominus q^{-1}\mathbf{V}_i \oplus \bigoplus_{e:s\to i} q^{-1}e^{-\mathfrak{m}_e}\mathbf{V}_s \oplus \bigoplus_{e:i\to t} e^{\mathfrak{m}_e}\mathbf{V}_t. \tag{84}$$

where q^{-1} and $e^{\pm\mathfrak{m}_e}$ denote the representations with respective characters.

We remind of the bundles $\mathbf{Q}$, $\mathbf{N}_i$, $\mathbf{K}_i$ (introduced in remark 3.14) over the instanton moduli space $\mathfrak{M}(\underline{n}, \underline{k})$. Furthermore, we define the virtual bundle $\mathbf{S}_i := \mathbf{N}_i - \wedge_{-1}\mathbf{Q}\mathbf{K}_i$.

Over the product $\mathfrak{M}(\underline{n},\underline{k}) \times \mathfrak{Q}(\underline{w},\underline{v})$ of the instanton moduli space and the quiver variety, we define the virtual bundle

$$\mathbf{G} := \bigoplus_{i\in\mathrm{Vert}_\gamma} (q\mathbf{S}_i^*\mathbf{C}_i \oplus \mathbf{M}_i^*\mathbf{V}_i) \tag{85}$$

(where the factors should be understood to mean their pullbacks by the projections $\pi_{\mathfrak{M}}, \pi_{\mathfrak{Q}} : \mathfrak{M}(\underline{n},\underline{k}) \times \mathfrak{Q}(\underline{w},\underline{v}) \to \mathfrak{M}(\underline{n},\underline{k}), \mathfrak{Q}(\underline{w},\underline{v})$.) For each fixed point $\underline{\boldsymbol{\lambda}} \in \mathfrak{M}(\underline{n},\underline{k})$, we consider the inclusion

$$\iota_{\underline{\boldsymbol{\lambda}}} : \mathfrak{Q}(\underline{w},\underline{v}) \to \mathfrak{M}(\underline{n},\underline{k}) \times \mathfrak{Q}(\underline{w},\underline{v}), \quad x \mapsto (\underline{\boldsymbol{\lambda}}, x).$$

Then the pullback $\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G}$ is a bundle over the quiver variety $\mathfrak{Q}(\underline{w},\underline{v})$.

Definition 4.10. *For a quiver gauge theory $(\gamma,\underline{n},\underline{m})$, its cohomology qq-character of highest weight $\underline{w} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$, with shifts $\underline{\sigma} = (\sigma_{i,p})_{i\in\mathrm{Vert}_\gamma, p=1,\dots,w_i}$, is the series*

$$\mathcal{X}_{\underline{w},\underline{\sigma}}(x)[\underline{\boldsymbol{\lambda}}] := \sum_{\underline{v}} \mathfrak{q}^{\underline{v}} \int_{\mathfrak{Q}(\underline{w},\underline{v})} c_{\varepsilon_2}^{T_{\underline{w}}\times T}(T\mathfrak{Q}(\underline{w},\underline{v}))\, c_x^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G}), \tag{86}$$

where $\underline{\boldsymbol{\lambda}}$ is a random $\underline{n}$-colored partition and the integrals are $T_{\underline{w}} \times T$-equivariant. The corresponding K-theory qq-character is

$$\mathcal{X}_{\underline{w},\underline{\tau}}^K(z)[\underline{\boldsymbol{\lambda}}] := \sum_{\underline{v}} \mathfrak{q}^{\underline{v}} \left(\frac{-q_1}{q_2}\right)^{-\dim_{\mathbb{C}}\mathfrak{Q}(\underline{w},\underline{v})} \int_{\mathfrak{Q}(\underline{w},\underline{v})} \mathrm{Td}^{T_{\underline{w}}\times T}(T\mathfrak{Q})\, \mathrm{Ch}^{T_{\underline{w}}\times T}(\wedge_{-q_2^{-1}}T\mathfrak{Q})\, c_z^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G}), \tag{87}$$

where $\underline{\tau}$ is identified with the exponentiated equivariant parameters of $T_{\underline{w}}$ and the integrals are $T_{\underline{w}} \times T$-equivariant.

There are some subtleties about this definition:

- As the quiver variety is in general non-compact, the integrals are to be understood as defined by equivariant localization: The compactness of the fixed point submanifold of $\mathfrak{Q}(\underline{w},\underline{v})$ is a special case of Nekrasov's compactness theorem for crossed quiver instantons [Nek17, Sec. 10.1.5].

- $c_{\varepsilon_2}^{T_{\underline{w}}\times T}$ is the equivariant Chern polynomial evaluated at $x = \varepsilon_2$. In the limit $\varepsilon_2 = 0$, it becomes the equivariant Euler class.

- $c_x^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G})$ is the equivariant Chern polynomial of the virtual bundle. For virtual bundles $E \ominus F$, Chern polynomials are defined as $\frac{c_x(E)}{c_x(F)}$. This poses the question of how to understand the reciprocal $1/c_x(F)$ and how to integrate it. As $c_x(F)$ is an equivariant characteristic class (a symmetric polynomial of equivariant formal Chern roots), over a fixed point set F it can be inverted in the localization of $H_T^\bullet(F)$ with respect to its differential-form-degree zero polynomial (cf. [GS13, Sec. 10.8]). The integral, which is $S(\mathfrak{t})$-linear, is uniquely extended to a $S(\mathfrak{t})$-linear map on this localization.

- Recall $\wedge_{-q_2^{-1}}E = \sum_{i=0}^{\mathrm{rk}\,E}(-q_2^{-1})^i \wedge^i E$. When $q_2 = 1$, this has $\mathrm{Ch}^{T_{\underline{w}}\times T}\wedge_{-1}E = \prod_i(1-e^{-r_i})$ in terms of the formal Chern roots. Thus in this limit, $\mathrm{Td}^{T_{\underline{w}}\times T}(T\mathfrak{Q})\,\mathrm{Ch}^{T_{\underline{w}}\times T}(\wedge_{-1}T\mathfrak{Q}) = \prod_i r_i = e^{T_{\underline{w}}\times T}(T\mathfrak{Q})$ is the equivariant Euler class.

- The virtual characteristic class $c_x^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G})$ can be expressed as a symmetric rational function of the involved bundles' roots:

$$\prod_{i\in\mathrm{Vert}_\gamma}\left(\prod_{v\in\mathrm{Roots}(W_i)}\mathcal{Y}_i(x+\varepsilon+v)\prod_{\phi\in\mathrm{Roots}(\mathbf{V}_i)}\frac{P_i(x+\phi)}{\mathcal{Y}_i(x+\phi)\mathcal{Y}_i(x+\varepsilon+\phi)}\right.$$
$$\left.\prod_{e:\mathrm{in}(e)=i}\prod_{\phi\in\mathrm{Roots}(\mathbf{V}_{\mathrm{out}(e)})}\mathcal{Y}_i(x+\mathfrak{m}_e+\varepsilon+\phi)\prod_{e:\mathrm{out}(e)=i}\prod_{\phi\in\mathrm{Roots}(\mathbf{V}_{\mathrm{in}(e)})}\mathcal{Y}_i(x-\mathfrak{m}_e+\phi)\right) \quad (88)$$

To see this, use $\mathcal{Y}_i(x)=c_x^T(S_i^*)$ and $P_i(x)=c_x^T(M_i^*)$. The analogous formula holds for $c_z^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G})$ in K-theory.

Some facts about qq-characters. Using the localization technique of [MNS00], the equivariant integrals defining the qq-characters can be transformed into a multi-dimensional contour integral. While the original contour integral given by Nekrasov [Nek16, p. 8.3.2] is incorrect, a corrected version can be found in [Kim21, equation 5.5.5] (this is for K-theory but carries over to cohomology). This implies that the qq-character is in fact a Laurent series in shifted $\mathcal{Y}_i$-functions and their derivatives. The contour integral is also a convenient way of checking that the qq-characters we computed in Section 4.2 are, in fact, the same as what we would get using definition 4.10. Furthermore, it is easy to see from the contour integral that qq-characters are independent of the coloring $\underline{n}$ (as Laurent series in the $\mathcal{Y}_i$-functions, not as functions on the Lie algebra).

The $\underline{\mathfrak{q}}^0$-term in the qq-characters (86), (87) is called the highest weight monomial. We give an explicit expression: It is the integral over the quiver variety $\mathfrak{Q}(\underline{w},0)$ which consists of just one point. As the bundles $\mathbf{V}_i$ are all zero, the bundle $\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G}$ is then simply the representation $qS_i[\underline{\boldsymbol{\lambda}}]^*W_i$ and

$$\int_{\mathfrak{Q}(\underline{w},0)}c_{\varepsilon_2}^{T_{\underline{w}}\times T}(T\mathfrak{Q}(\underline{w},\underline{v}))\,c_x^{T_{\underline{w}}\times T}(\iota_{\underline{\boldsymbol{\lambda}}}^*\mathbf{G})=c_x^{T_{\underline{w}}\times T}(qS_i[\underline{\boldsymbol{\lambda}}]^*W_i).$$

Using $\mathcal{Y}_i(x)=c_x^{T_{\underline{w}}\times T}(S_i[\underline{\boldsymbol{\lambda}}]^*)$,

$$c_x^{T_{\underline{w}}\times T}(qS_i[\underline{\boldsymbol{\lambda}}]^*W_i)=\prod_{i\in\mathrm{Vert}_\gamma}\prod_{p=1}^{w_i}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p}).$$

Thus, the qq-character always has the form

$$\mathcal{X}_{\underline{w},\underline{\sigma}}(x)=\prod_{i\in\mathrm{Vert}_\gamma}\prod_{p=1}^{w_i}\mathcal{Y}_i(x+\varepsilon+\sigma_{i,p})+\mathcal{O}(\mathfrak{q}).$$

The same calculation for the K-theory qq-character yields

$$\mathcal{X}^K_{\underline{w},\underline{\tau}}(z)=\prod_{i\in\mathrm{Vert}_\gamma}\prod_{p=1}^{w_i}\mathcal{Y}_i(q\tau_{i,p}z)+\mathcal{O}(\mathfrak{q}).$$

The quiver variety has dimension $\dim_{\mathbb{C}}\mathfrak{Q}(\underline{w},\underline{v})=2\underline{v}\cdot\underline{w}-2\langle\underline{v},\underline{v}\rangle$, where the Euler form of the quiver is $\langle\underline{v},\underline{w}\rangle=\sum_{i\in\mathrm{Vert}_\gamma}v_iw_i-\sum_{e\in\mathrm{Edges}_\gamma}v_{\mathrm{in}(e)}w_{\mathrm{out}(e)}$ (this is asymmetric). The quiver variety is empty when this dimension is negative. For quivers of type ADE, the quiver variety is empty when $\underline{v} > \underline{w}$ (entry-wise). This implies that qq-characters of ADE quiver gauge theories have no terms with coefficient $\underline{\mathfrak{q}}^{\underline{v}}$ for $\underline{v} > \underline{w}$, meaning they have only finitely many terms.

The Euler form's symmetrization is given by the quiver's Cartan matrix (cf. [Kir16, eq. 1.25]). Thus, for affine Dynkin quivers (i.e. type $\hat{A}\hat{D}\hat{E}$), there is a non-zero $\underline{v} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ such that $\langle \underline{v}, \underline{v} \rangle = 0$. This means that for $\hat{A}\hat{D}\hat{E}$ quiver gauge theories, the qq-characters have infinitely many terms.

Nekrasov's theorem. As the measure $\mu_{\underline{\boldsymbol{\lambda}}}$ is defined by instanton counting, the expectation of the qq-character may be given as a double integral:

$$\begin{aligned}\langle \mathcal{X}_{\underline{w},\underline{\sigma}}(x) \rangle &= \sum_{\underline{k}} \underline{\mathfrak{q}}^{\underline{k}} \sum_{\underline{\boldsymbol{\lambda}} \in \mathfrak{P}(\underline{n},\underline{k})} Z_{\underline{\boldsymbol{\lambda}}} \left(\sum_{\underline{v}} \underline{\mathfrak{q}}^{\underline{v}} \int_{\mathfrak{Q}(\underline{w},\underline{v})} c_{\varepsilon_2}^{T_{\underline{w}} \times T}(T\mathfrak{Q}(\underline{w},\underline{v}))\, c_x^{T_{\underline{w}} \times T}(\iota_{\underline{\boldsymbol{\lambda}}}^* \mathbf{G}) \right) \\ &= \sum_{\underline{k},\underline{v}} \underline{\mathfrak{q}}^{\underline{k}+\underline{v}} \int_{\mathfrak{M}(\underline{n},\underline{k}) \times \mathfrak{Q}(\underline{w},\underline{v})} \epsilon^T(\mathbf{B})\, c_{\varepsilon_2}^{T_{\underline{w}} \times T}(T\mathfrak{Q}(\underline{w},\underline{v}))\, c_x^{T_{\underline{w}} \times T}(\mathbf{G}) \end{aligned} \tag{89}$$

We state Nekrasov's main theorem. His statement of this in [Nek16, Sec. 6.1] is without the assumption $\underline{\beta} \leq 0$. However, as we saw in preceding examples, in cases where some $\beta_i > 0$, the polynomial degree of the qq-characters' expectations does not satisfy his claim. We expect that an extension of the theorem to general $\underline{\beta}$ is possible; the expectations of ADE theory qq-characters should still be polynomial, albeit of different degree, and those of $\hat{A}\hat{D}\hat{E}$ theories would at least be non-singular.

Conjecture 4.11. *Let $(\gamma, \underline{n}, \underline{m})$ be a quiver gauge theory with beta function $\underline{\beta} \leq 0$ (in particular, γ is of type ADE or $\hat{A}\hat{D}\hat{E}$, cf. below (40)). Then the expectations of all cohomology qq-characters $\mathcal{X}_{\underline{w},\underline{\sigma}}(x)$ of the quiver gauge theory are polynomials of degree $\underline{w} \cdot \underline{n}$.*

We state this as a conjecture because we are unsure if Nekrasov's proof holds in the ADE case and not only for $\hat{A}\hat{D}\hat{E}$ quivers. His proof [Nek18] is based on regarding the expectation $\langle \mathcal{X}_{\underline{w},\underline{\sigma}}(x) \rangle$ as the partition function of an orbifolded crossed gauge origami model (the abelian $\times$ ALE case), defined as a generating function of equivariant integrals over the product of the moduli space of spiked instantons (in the crossed case) and the quiver variety, much like (89). The compactness theorem [Nek17] then implies that the partition function has no singularities in certain equivariant parameters identified with x.

It appears that the proof depends on γ being affine Dynkin. In the ADE case, it may be feasible to prove that the qq-characters are generated by an adapted FM algorithm, and, using incremental desingularization, that this algorithm always produces polynomial expectation.

Nekrasov's proof should carry over to K-theory qq-characters. In this case, the expectations would be polynomials of degree $\underline{w} \cdot \underline{n}$ in z^{-1} rather than in x.

Example computation. We compute the fundamental qq-character of $(\hat{A}_0, \underline{n}, 0)$ quiver gauge theory using the geometric formula (86). The quiver variety $\mathfrak{Q}(w, v)$ of the $\hat{A}_0$ quiver is the same as the resolved instanton moduli space $\mathfrak{M}(w, v)$. As outlined in the beginning of the section, the torus $T_w \times T \cong (\mathbb{C}^\times)^w \times (\mathbb{C}^\times)^n \times \mathbb{C}^\times \times (\mathbb{C}^\times)^2$ acts on the quiver variety or moduli space by

$$e^{(\sigma,\mathfrak{a},\mathfrak{m},\vec{\varepsilon})} : (B_1, B_2, I, J) \mapsto (e^{\mathfrak{m}} B_1, e^{-\mathfrak{m}-\varepsilon} B_2, I e^{\sigma}, e^{\varepsilon} e^{-\sigma} J), \tag{90}$$

where $\mathfrak{m}$ is the edge mass. This coincides with the previous action (19), (20) on $\mathfrak{M}(w,v)$ when one substitutes $\mathfrak{a} \to \sigma$ and $\varepsilon_1 \to \mathfrak{m}$, $\varepsilon_2 \to -\mathfrak{m} - \varepsilon$.

From instanton counting, we know all fixed points (they are parametrized by partitions) and the structures of the tangent isotropy representations. While the torus action is now modified compared to Chapter 3, the fixed points are the same and we must only replace $\varepsilon_1 \to \mathfrak{m}$, $\varepsilon_2 \to -\mathfrak{m} - \varepsilon$ to get the correct characters of representations. Localization yields

$$\int_{\mathfrak{M}(w,v)} c^T_{\varepsilon_2}(T\mathfrak{M}(w,v))\, c^T_x(\mathbf{G}) = \sum_{\underline{\lambda} \in \mathfrak{P}(w,v)} \frac{c^T_{\varepsilon_2}(T_{\underline{\lambda}}\mathfrak{M}(w,v))\, c^T_x(\mathbf{G}|_{\underline{\lambda}})}{\epsilon^T(T_{\underline{\lambda}}\mathfrak{M}(w,v))}.$$

For the fundamental qq-character, $w = 1$. $\mathfrak{P}(1,v)$ is the set of all partitions of size v (1-colored partitions). Then the qq-character is (renaming v to k to match the notation in Chapter 3)

$$\mathcal{X}_{\underline{\delta}_1,0}(x) = \sum_{k=0}^{\infty} \mathfrak{q}^k \sum_{\lambda \in \mathfrak{P}(1,k)} \frac{c^T_{\varepsilon_2}(T_\lambda\mathfrak{M}(1,k))\, c^T_x(\mathbf{G}|_\lambda)}{\epsilon^T(T_\lambda\mathfrak{M}(1,k))}.$$

The shift parameter σ is redundant for fundamental qq-characters (there is only on $\mathcal{Y}$-function in the highest weight monomial), hence we set it to zero. By (84), the virtual representation $\mathbf{G}|_\lambda$ is isomorphic to

$$\mathbf{G}|_\lambda \cong \mathbf{N}|_\lambda - \mathbf{K}|_\lambda - e^{-\varepsilon}\mathbf{K}|_\lambda + e^{-\mathfrak{m}-\varepsilon}\mathbf{K}|_\lambda + e^{\mathfrak{m}}\mathbf{K}|_\lambda,$$

where $\mathbf{N}, \mathbf{K} \to \mathfrak{M}(1,k)$ are the framing and instanton bundles from remark 3.14. By the computations in Section 3.2.2 and the identifications of equivariant parameters $\underline{\mathfrak{a}} \to \underline{\sigma}, \varepsilon_1 \to \mathfrak{m}, \varepsilon_2 \to -\mathfrak{m} - \varepsilon$, the characters are

$$\mathrm{Ch}^T(\mathbf{N}|_\lambda) = e^{\sigma} = 1, \qquad \mathrm{Ch}^T(\mathbf{K}|_\lambda) = \sum_{\square \in \lambda} \mathrm{e}^{\mathfrak{k}_\square}$$

where $\mathfrak{k}_\square := (s_1 - 1)\mathfrak{m} + (s_2 - 1)(-\mathfrak{m} - \varepsilon)$ is the content relative to the weights $(\mathfrak{m}, -\mathfrak{m} - \varepsilon)$. This, together with (88), implies

$$c^T_x(\mathbf{G}|_\lambda) = \mathcal{Y}(x+\varepsilon) \prod_{\square \in \lambda} \frac{\mathcal{Y}(x + \mathfrak{k}_\square - \mathfrak{m})\mathcal{Y}(x + \mathfrak{k}_\square + \varepsilon + \mathfrak{m})}{\mathcal{Y}(x + \mathfrak{k}_\square)\mathcal{Y}(x + \mathfrak{k}_\square + \varepsilon)}. \tag{91}$$

From Chapter 3,

$$\frac{1}{\epsilon^T(T_\lambda\mathfrak{M}(1,k))} = \prod_{\square \in \lambda} \frac{1}{-\mathfrak{k}_\square(\mathfrak{k}_\square + \varepsilon)} \prod_{\square,\square' \in \lambda} S(\mathfrak{k}_{\square'} - \mathfrak{k}_\square)^{-1}, \tag{92}$$

and similarly

$$c^T_{\varepsilon_2}(T_\lambda\mathfrak{M}(1,k)) = \prod_{\square \in \lambda} (-\mathfrak{k}_\square + \varepsilon_2)(\mathfrak{k}_\square + \varepsilon + \varepsilon_2) \prod_{\square,\square' \in \lambda} S(\mathfrak{k}_{\square'} - \mathfrak{k}_\square + \varepsilon_2). \tag{93}$$

The product of (91), (92), (93) is telescopic (cf. [Kim21, equation B.2.4]), simplifying the end result:

$$\mathcal{X}_{\underline{\delta}_1,0}(x) = \mathcal{Y}(x+\varepsilon) \sum_\lambda \mathfrak{q}^{|\lambda|} \prod_{\square \in \lambda} S(h_\square \mathfrak{m} + a_\square \varepsilon) \prod_{\square \in \lambda} \frac{\mathcal{Y}(x + \mathfrak{k}_\square - \mathfrak{m})\mathcal{Y}(x + \mathfrak{k}_\square + \varepsilon + \mathfrak{m})}{\mathcal{Y}(x + \mathfrak{k}_\square)\mathcal{Y}(x + \mathfrak{k}_\square + \varepsilon)}$$

where $a_\square = \lambda_{s_1} - s_2$ is the arm length and $h_\square = \lambda_{s_1} - s_2 + \check{\lambda}_{s_2} - s_1 + 1$ the hook length. We see that the fundamental qq-character of $\hat{A}_0$ theory is infinite and each Laurent monomial of $\mathcal{Y}$-functions has one more $\mathcal{Y}$-function in the numerator than in the denominator.

4.5 Representation theoretic interpretation of qq-characters

Throughout, all Lie algebras are complex and all representations are finite-dimensional.

4.5.1 Highest weight representations

We start off with an overview of the theory of complex simple Lie algebras $\mathfrak{g}$, as in e.g. [FH13]. Complex simple Lie algebras are classified by their Dynkin diagrams. In the Dynkin diagram γ (which is not a quiver), each vertex i corresponds to a simple root $\alpha_i \in \mathfrak{h}^\vee$ (where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$), a simple coroot $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ and the fundamental weight $\Lambda_i \in \mathfrak{h}^\vee$ which satisfies $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. The fundamental weights span the weight lattice

$$P = \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_N \subset \mathfrak{h}^\vee$$

which contains the root lattice Q spanned by the simple roots. All weights of all representations of $\mathfrak{g}$ are contained in P. A partial ordering on the weight lattice is given by

$$\lambda \geq \mu \qquad \Longleftrightarrow \qquad \lambda = \mu + \sum_{i \in \mathrm{Vert}_\gamma} c_i \Lambda_i \text{ for some } c_i \geq 0.$$

A weight $\lambda \in P$ is called dominant if $\lambda \geq 0$. The highest weight of a representation V of $\mathfrak{g}$ is a weight λ of V that dominates all other weights, i.e. $\lambda \geq \mu$ for all $\mu \in \mathrm{Weights}(V)$.

Theorem 4.12 ([Car13]). *An irreducible representation of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ is determined up to isomorphism by its highest weight, which is a dominant element of P. Furthermore, each dominant weight $\lambda \in P$ is the highest weight of an irreducible representation of $\mathfrak{g}$.*

A representation V's character is

$$\chi_V = \sum_{\lambda \in P} m(\lambda) e^\lambda$$

where $m(\lambda)$ is the multiplicity of the weight. This motivates defining the exponentiated fundamental weights $y_i = e^{\Lambda_i}$. These play a similar role for characters of representations as the $\mathcal{Y}_i$-observables play for qq-characters. A weight λ of the representation V is expressed as an integer-coefficient linear combination of the fundamental weights, $\lambda = \sum_i \lambda_i \Lambda_i$. Consequently, the associated exponentiated weight is a Laurent monomial in the y_i:

$$e^\lambda = e^{\sum_i \lambda_i \Lambda_i} = \prod_{i \in \mathrm{Vert}_\gamma} y_i^{\lambda_i}, \qquad \lambda_i \in \mathbb{Z}$$

This makes it clear that the character of V is in fact a Laurent polynomial in the exponentiated fundamental weights,

$$\chi_V = \sum_{\lambda \in P} m(\lambda) \prod_{i \in \mathrm{Vert}_\gamma} y_i^{\lambda_i},$$

much like how the qq-character is a Laurent polynomial or series in the shifted $\mathcal{Y}_i$-observables (and their derivatives).

For a highest weight representation $V(\lambda)$, we now outline the construction of its character from the highest weight λ. We will see that this process is similar to constructing a qq-character from the highest weight monomial by repeated desingularization. It suffices to know all weights of $V(\lambda)$ and their multiplicities. Any weight μ of $V(\lambda)$ must be $\mu \leq \lambda$. To generate weights lower than λ, we use the Weyl reflections: For each vertex i of the Dynkin diagram, the associated simple root α_i defines a reflection

$$s_i : \mathfrak{h}^\vee \to \mathfrak{h}^\vee, \quad \mu \mapsto \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$$

along the hyperplane orthogonal to it. This is called the Weyl reflection of vertex i. The Coxeter group generated by the Weyl reflection is called the Weyl group W. Using the defining relation $\langle \Lambda_j, \alpha_i^\vee \rangle = \delta_{ij}$, we see that the Weyl reflections act on fundamental weights by

$$s_i(\Lambda_j) = \Lambda_j - \delta_{ij}\alpha_i.$$

As the s_i are linear, this extends to elements $\mu = \sum_j \mu_j \Lambda_j$ of the weight lattice P,

$$s_i(\mu) = \mu - \mu_i \alpha_i,$$

meaning that the weights lower than λ are all obtained from λ by applying Weyl reflections. The multiplicities of the new weights can be computed using Freudenthal's recursion formula. We also define the Weyl reflections on the exponentiated weights $e^\mu = \prod_j y_j^{\mu_j} \in \mathbb{C}[y_j^{\pm 1}]_{j \in \mathrm{Vert}_\gamma}$, by $s_i(e^\mu) := e^{s_i(\mu)}$. This way, the Weyl reflections are ring homomorphisms. Then,

$$s_i \left(\prod_{j \in \mathrm{Vert}_\gamma} y_j^{\mu_j} \right) = A_i^{-\mu_i} \prod_{j \in \mathrm{Vert}_\gamma} y_j^{\mu_j}$$

where we define $A_i = e^{\alpha_i}$, the exponentiated simple roots. Note the similarity of $s_i(y_i) = A_i^{-1} y_i$ to the iWeyl reflection (66) in the construction of qq-characters.

It can be seen from e.g. Freudenthal's recursion formula that all weights in $V(\lambda)$ are related by addition or subtraction of simple roots. Starting from the highest weight λ, with associated monomial

$$e^\lambda = \prod_{i \in \mathrm{Vert}_\gamma} y_i^{\lambda_i} = y_{i_1} \dots y_{i_{|\lambda|}}, \qquad |\lambda| = \sum_{i \in \mathrm{Vert}_\gamma} \lambda_i,$$

all lower weights occuring in the highest weight representation $V(\lambda)$ are thus obtained by repeatedly applying Weyl reflections to factors y_{i_k}, i.e. all Laurent monomials in the character are of the form

$$s_{j_1}^{k_1}(y_{i_1}) \dots s_{j_{|\lambda|}}^{k_{|\lambda|}}(y_{i_{|\lambda|}}),$$

where $k_1, \dots, k_{|\lambda|} = 0, 1$. This is analogous to generating the qq-character from the highest weight monomial

$$\mathcal{Y}_{i_1}(x + \varepsilon + \sigma_{i_1,1}) \dots \mathcal{Y}_{i_{|\underline{w}|}}(x + \varepsilon + \sigma_{i_{|\underline{w}|}, w_{i_{|\underline{w}|}}})$$

by repeatedly applying iWeyl reflections to the individual $\mathcal{Y}$-functions in the numerators. We should, however, point out that the analog of the Frenkel–Mukhin algorithm for ordinary characters fails most of the time.

Once all weights are found using Weyl reflections, their multiplicities can be computed recursively using Freudenthal's recursion formula.

4.5.2 q-characters

We now provide some results and conjectures about the relationship between quiver gauge theory qq-characters in the Nekrasov–Shatashvili limit $\varepsilon_2 \to 0$, and the q-characters of quantum group representations. We only conjecture the precise relationship, to resolve this we would need a more thorough analysis of the quiver varieties' fixed point submanifolds under different torus actions.

As the quiver variety $\mathfrak{Q}(\underline{w}, \underline{v})$ is a quotient by $U(\underline{v})$, under the $T_{\underline{w}} \times T$-action, the fixed point set is stratified

$$\mathfrak{Q}(\underline{w}, \underline{v})^{T_{\underline{w}} \times T} = \bigsqcup_{\eta} F(\eta),$$

where the union goes over $U(\underline{v})$-conjugacy classes of homomorphisms $\eta : T_{\underline{w}} \times T \to U(\underline{v})$, and $F(\eta)$ is the associated fixed point submanifold. This is analogous to the fixed points on the resolved instanton moduli space being associated with homomorphisms $\phi : T \to U(k)$. The fixed point sets $F(\eta)$ are compact due to Nekrasov's compactness theorem for crossed quiver instantons [Nek17, Sec. 10.1.5].

Lemma 4.13. *Let $(\gamma, \underline{n}, \underline{m})$ be a quiver gauge theory. Then, in the limit $\varepsilon_2 = 0$ (Nekrasov–Shatashvili limit), all cohomology qq-characters are Laurent series in shifted $\mathcal{Y}_i$-observables and matter polynomials P_i, not involving their derivatives.*

Proof. The basic observation is that, for $\varepsilon_2 = 0$, the Chern polynomial $c^T_{\varepsilon_2}(T\mathfrak{Q}(\underline{w}, \underline{v}))$ is the equivariant Euler class $\epsilon^{T_{\underline{w}} \times T}(T\mathfrak{Q}(\underline{w}, \underline{v}))$. In the K-theory case, the same holds for $\mathrm{Td}^{T_{\underline{w}} \times T}(T\mathfrak{Q})\, \mathrm{Ch}^{T_{\underline{w}} \times T}(\wedge_{-q_2^{-1}} T\mathfrak{Q})$. Our proof is for the cohomology case, for K-theory it would be analogous. In the $\varepsilon_2 = 0$ limit, the qq-character is

$$\mathcal{X}_{\underline{w}, \underline{\sigma}}(x)[\underline{\boldsymbol{\lambda}}] = \sum_{\underline{v}} \mathfrak{q}^{\underline{v}} \int_{\mathfrak{Q}(\underline{w}, \underline{v})} \epsilon^{T_{\underline{w}} \times T}(T\mathfrak{Q}(\underline{w}, \underline{v}))\, c_x^{T_{\underline{w}} \times T}(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G}).$$

Now we perform localization to fixed point sets $F(\eta)$:

$$\int_{\mathfrak{Q}(\underline{w}, \underline{v})} \epsilon^{T_{\underline{w}} \times T}(T\mathfrak{Q}(\underline{w}, \underline{v}))\, c_x^{T_{\underline{w}} \times T}(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G}) = \sum_{\eta} \int_{F(\eta)} \frac{\epsilon^{T_{\underline{w}} \times T}(\iota^*_{F(\eta)} T\mathfrak{Q})\, \iota^*_{F(\eta)} c_x^{T_{\underline{w}} \times T}(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G})}{\epsilon^{T_{\underline{w}} \times T}(NF(\eta))}$$

Using $NF(\eta) = \iota^*_{F(\eta)} T\mathfrak{Q} / TF(\eta)$, this simplifies to

$$\sum_{\eta} \int_{F(\eta)} \epsilon^{T_{\underline{w}} \times T}(TF(\eta))\, \iota^*_{F(\eta)} c_x^{T_{\underline{w}} \times T}(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G}).$$

Now, as the action on F is trivial, in the Cartan model the equivariant Euler class $\epsilon^{T_{\underline{w}} \times T}(TF(\eta))$ is equal to the non-equivariant Euler class $\epsilon(TF(\eta))$:

$$\sum_{\eta} \int_{F(\eta)} \epsilon(TF(\eta))\, \iota^*_{F(\eta)} c_x^{T_{\underline{w}} \times T}(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G})$$

In this integral, because $\epsilon(TF(\eta))$ has degree $\dim(F(\eta))$, the equivariant form $\iota^*_{F(\eta)} c_x(\iota^*_{\underline{\boldsymbol{\lambda}}} \mathbf{G})$ contributes only with its component that has differential form degree 0, i.e. the polynomial (rational function) part. By the identification of the equivariant Chern roots with weights (cf. beneath Lemma 2.13), this is the rational function (88) but for the roots we enter the weights of V_i, W_i, which depend on the particular homomorphism η:

$$\mathcal{X}_{\underline{w},\underline{\sigma}}(x) = \sum_{\underline{v}} \mathfrak{q}^{\underline{v}} \sum_{\eta} \chi(F(\eta)) \prod_{i \in \mathrm{Vert}_\gamma} Q_i(x) \tag{94}$$

with

$$Q_i(x) = \prod_{p=1}^{w_i} \mathcal{Y}_i(x+\varepsilon+\sigma_{i,p}) \prod_{t=1}^{v_i} \frac{P_i(x+d_{i,t}(\eta))}{\mathcal{Y}_i(x+d_{i,t}(\eta))\mathcal{Y}_i(x+\varepsilon+d_{i,t}(\eta))}$$
$$\prod_{e:\mathrm{in}(e)=i} \prod_{t=1}^{v_{\mathrm{out}(e)}} \mathcal{Y}_i(x+\mathfrak{m}_e+\varepsilon+d_{\mathrm{out}(e),t}(\eta)) \prod_{e:\mathrm{out}(e)=i} \prod_{t=1}^{v_{\mathrm{in}(e)}} \mathcal{Y}_i(x-\mathfrak{m}_e+d_{\mathrm{in}(e),t}(\eta))$$

where $d_{i,t}(\eta)$ are the weights of the representation on V_i induced by $\eta : T_{\underline{w}} \times T \to U(\underline{v})$. □

Alternatively, (94) also holds with

$$Q_i(x) = \prod_{p=1}^{w_i} \mathcal{Y}_i(x+\varepsilon+\sigma_{i,p}) \prod_{t=1}^{v_i} P_i(x+d_{i,t}(\eta))$$
$$\prod_{t=1}^{v_i} \frac{\prod_{e:\mathrm{out}(e)=i} \mathcal{Y}_{\mathrm{in}(e)}(x+\mathfrak{m}_e+\varepsilon+d_{i,t}(\eta)) \prod_{e:\mathrm{in}(e)=i} \mathcal{Y}_{\mathrm{out}(e)}(x-\mathfrak{m}_e+d_{i,t}(\eta))}{\mathcal{Y}_i(x+d_{i,t}(\eta))\mathcal{Y}_i(x+\varepsilon+d_{i,t}(\eta))}.$$

Note that the second row is equal to $\prod_{t=1}^{v_i} A_i(x+d_{i,t}(\eta))^{-1}$, where $A_i(x)$ was defined in the context of iWeyl reflection in (66).

In the $\varepsilon_2 = 0$ (equivalently $q_2 = 1$) limit, qq-characters look a lot like q-characters, which are also Laurent series in formal variables $Y_{i,x}$. The q-character is a generalized character defined by Knight [Kni95] for Yangians, and by Frenkel and Reshetikhin [FR99] for quantum affine algebras. Both Yangians and quantum affine algebras are quantum groups (which aren't actually groups at all but rather Hopf algebras). The reason we need both of these quantum groups is that the Yangian q-characters relate to cohomology qq-characters while the q-characters of quantum affine algebras relate to K-theory qq-characters.

Yangians. Given a simple Lie algebra $\mathfrak{g}$, its polynomial loop algebra is $\mathfrak{g} \otimes \mathbb{C}[t]$ (with commutation relations as in [CP95b, Sec. 12.1]). The Yangian of $\mathfrak{g}$, denoted $Y(\mathfrak{g})$, is a certain deformation of the universal enveloping algebra $U(\mathfrak{g} \otimes \mathbb{C}[t])$. Precisely, the commutation relations of certain generators of $U(\mathfrak{g} \otimes \mathbb{C}[t])$ are modified by use of a deformation parameter $\varepsilon \in \mathbb{C}$. The Yangian $Y(\mathfrak{g})$ depends on this parameter, even if it doesn't show in the notation. In the limit where $\varepsilon = 0$, the Yangian reduces to the polynomial loop algebra's universal enveloping algebra; $Y(\mathfrak{g}) = U(\mathfrak{g} \otimes \mathbb{C}[t])$. A full definition of Yangians, including the precise commutation relations, can be found in [CP95b, Sec. 12.1].

Quantum affine algebras. The affinization of the simple Lie algebra $\mathfrak{g}$ is

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is the loop algebra, c a central element, and d a derivation of $\mathfrak{g}$. For the commutation relations, see [CP95b, Sec. 12.2]. This affine Lie algebra has a universal enveloping algebra $U(\hat{\mathfrak{g}})$. By introducing a deformation parameter q, Drinfeld [Dri85] and Jimbo [Jim85] deformed $U(\hat{\mathfrak{g}})$ to $U_q(\hat{\mathfrak{g}})$ (with the goal of studying and providing solutions to the quantum Yang–Baxter equation), the quantum affine algebra associated with $\mathfrak{g}$. Precisely, the parameter is used to alter the relations between the generators of $U(\hat{\mathfrak{g}})$. In the case $q = 1$, the relations are unchanged, i.e. $U_1(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}})$.

The representation theories of the Yangian and the quantum affine algebra are very similar. This is because the Yangian is in a sense a degeneration of the quantum loop algebra $U_q(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$ [GM12].

The q-characters. In order to study the finite-dimensional representations of the Hopf algebras $Y(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$, first Knight [Kni95] and later Frenkel and Reshetikhin [FR99] introduced the q-characters. Just as the ordinary character is an injective homomorphism $\chi : \operatorname{Rep} \mathfrak{g} \to \mathbb{Z}[y_i^{\pm 1}]_{i \in \mathrm{Vert}_\gamma}$, the q-characters are injective ring homomorphisms

$$\chi_q : Y(\mathfrak{g}) \to \mathbb{Z}[Y_{i,\sigma_i}^{\pm 1}]_{i \in \mathrm{Vert}_\gamma, \sigma_i \in \mathbb{C}}, \qquad \chi_q : \operatorname{Rep} U_q(\hat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,\tau_i}^{\pm 1}]_{i \in \mathrm{Vert}_\gamma, \dots, n, \tau_i \in \mathbb{C}^\times},$$

where Rep is the Grothendieck ring of the category of finite-dimensional representations. The q-character χ_q was originally defined by use of the universal R-matrix of $U_q(\hat{\mathfrak{g}})$ [FR99]. Using his quiver varieties, Nakajima [Nak01] studied the representation theory of $U_q(\hat{\mathfrak{g}})$ for when $\mathfrak{g}$ is of type ADE. In particular, he gave a geometric definition of q-characters, which we will try to relate to qq-characters. First, we should mention some basic facts about the finite-dimensional representation theory of quantum groups.

Drinfeld [Dri87] defined a family of finite-dimensional irreducible $Y(\mathfrak{g})$-modules $\{U_i(\sigma)\}_{i \in \mathrm{Vert}_\gamma, a \in \mathbb{C}}$, called the fundamental $Y(\mathfrak{g})$-modules, that play a central role in the representation theory of quantum groups:

Lemma 4.14 ([CP95a], Cor. 3.6)**.** *Every finite-dimensional irreducible $Y(\mathfrak{g})$-module U is a subquotient (a quotient of a submodule) of a tensor product of fundamental $Y(\mathfrak{g})$-modules*

$$U(\underline{\sigma}) \coloneqq \bigotimes_{i \in \mathrm{Vert}_\gamma} \bigotimes_{p=1}^{w_i} U_i(\sigma_{i,p})$$

and the parameters $\sigma_{i,p} \in \mathbb{C}$ are uniquely determined (up to permutation) by U.

Fundamental modules also exist for the quantum group $U_q(\hat{\mathfrak{g}})$, we denote them by $V_i(\tau)$ (in this case $\tau \in \mathbb{C}^\times$ rather than $\sigma \in \mathbb{C}$ as for the Yangian) as well, and the analogous statement to Lemma 4.14 holds in this setting too.

The continuous parameter σ in the formal variables $Y_{i,\sigma}$ may initially seem puzzling. After all, simple Lie algebras don't need this extra parameter for their characters. Its presence for the Yangian (and that of τ for the quantum affine algebra) is made essentially due to the variable t in $\mathfrak{g} \otimes \mathbb{C}[t]$. Irreducible representations of $\mathfrak{g} \otimes \mathbb{C}[t]$ are exactly the pullbacks of irreducible $\mathfrak{g}$-representations by evaluation homomorphisms $\mathrm{ev}_\sigma : \mathfrak{g} \otimes \mathbb{C}[t] \to \mathfrak{g}$, $(X, p(t)) \mapsto p(\sigma)X$, of which one exists for each $\sigma \in \mathbb{C}$.

We give a geometric definition of the q-character based on Nakajima's work [Nak01], originally in the context of quantum affine algebras but extended to Yangians by [Var00]. For this, we must restrict attention to simple Lie algebras of type ADE (otherwise it is unclear how to associate a quiver to the Lie algebra). The directions of the edges in the Dynkin diagram do not matter for the definition.

The group $G_{\underline{w}}$ acts on the quiver varieties $\mathfrak{Q}(\underline{w}, \underline{v})$ as outlined in Section 4.4. We also need the $\mathbb{C}^\times$-action on $\mathfrak{Q}(\underline{w}, \underline{v})$ defined by

$$u \cdot (B_{e,+}, B_{e,-}, I_i, J_i) \coloneqq (uB_{e,+}, uB_{e,-}, uI_i, uJ_i)$$

(this action is more complicated for non-ADE quivers). The parameters $\sigma_{i,p}$ ($i \in \mathrm{Vert}_\gamma$, $p = 1, \dots, w_i$) and ε (resp. $\tau_{i,p}$ and q) define a semisimple element $(e^{\underline{\sigma}}, e^{\varepsilon})$ (resp. $(\underline{\tau}, q)$) of $G_{\underline{w}} \times \mathbb{C}^\times$. We define the subgroup A of $G_{\underline{w}} \times \mathbb{C}^\times$ as the Zariski closure of $\{(e^{\underline{\sigma}}, e^{\varepsilon})^j \mid j \in \mathbb{Z}\}$. The fixed point set is the union

$$\mathfrak{Q}(\underline{w}, \underline{v})^A = \bigsqcup_\rho E(\rho)$$

where ρ are $G_{\underline{v}}$-conjugacy classes of homomorphisms $A \to G_{\underline{v}}$ and $E(\rho)$ the corresponding fixed point submanifold of $\mathfrak{Q}(\underline{w}, \underline{v})$ ($E(\rho)$ is connected when q is not a root of unity). Each ρ defines a representation of A on each V_i and we denote the eigenvalues of $\underline{\sigma}$ by $\mathfrak{c}_{i,t}(\rho)$ and those of $\underline{\tau}$ by $c_{i,t}(\rho)$ (counted with multiplicities; $t = 1, \dots, v_i$). These eigenvalues $\mathfrak{c}_{i,t}$ lie in $\cup_p (\sigma_{i,p} + \mathbb{Z}\varepsilon)$ (resp. $c_{i,t} \in \cup_p \tau_{i,p} q^{\mathbb{Z}}$.)

Definition 4.15. *Let $\mathfrak{g}$ be a simple Lie algebra of type ADE. Suppose a $Y(\mathfrak{g})$-module U (respectively $U_q(\mathfrak{g})$-module V) is an irreducible subquotient of $U(\underline{\sigma})$ (respectively $V(\underline{\tau})$). Its q-character is*

$$\chi_q(V) \coloneqq \sum_{\underline{v}} \sum_\rho \chi(E(\rho)) \prod_{i \in \mathrm{Vert}_\gamma} \left(\prod_{p=1}^{w_i} Y_{i,\sigma_{i,p}} \prod_{t=1}^{v_i} A^{-1}_{i,\mathfrak{c}_{i,t}(\rho) - \varepsilon} \right) \tag{95}$$

(respectively with $q^{-1} c_{i,t}(\rho)$) where

$$A_{i,x-\varepsilon} \coloneqq Y_{i,x} Y_{i,x-2\varepsilon} \prod_{e:\mathrm{in}(e)=i} Y^{-1}_{\mathrm{out}(e),x-\varepsilon} \prod_{e:\mathrm{out}(e)=i} Y^{-1}_{\mathrm{in}(e),x-\varepsilon} \tag{96}$$

(and $A_{i,q^{-1}z}$ is defined analogously for $U_q(\hat{\mathfrak{g}})$) and the Euler characteristic χ is defined in Borel–Moore homology.

For example (cf. [FR99, Sec. 5.4]), the q-character of the fundamental $Y(\hat{\mathfrak{sl}}_2)$-module (respectively $U_q(\hat{\mathfrak{sl}}_2)$-module) ($A_1$ diagram) is

$$\chi_q(U_1(\sigma)) = Y_{1,\sigma} + \frac{1}{Y_{1,\sigma+2\varepsilon}}, \qquad \chi_q(V_1(\tau)) = Y_{1,\tau} + \frac{1}{Y_{1,\tau q^2}} \tag{97}$$

and that of the first fundamental $Y(\hat{\mathfrak{sl}}_3)$-module (respectively $U_q(\hat{\mathfrak{sl}}_3)$-module) ($A_2$ diagram) is

$$\chi_q(U_1(\sigma)) = Y_{1,\sigma} + \frac{Y_{2,\sigma+\varepsilon}}{Y_{1,\sigma+2\varepsilon}} + \frac{1}{Y_{2,\sigma+3\varepsilon}}, \qquad \chi_q(V_1(\tau)) = Y_{1,\tau} + \frac{Y_{2,\tau q}}{Y_{1,\tau q^2}} + \frac{1}{Y_{2,\tau q^3}}.$$

In the literature it is frequently stated that the $\varepsilon_2 \to 0$ or $q_2 \to 1$ limit of the qq-character is the q-character. It is more subtle than that:

- Given a simple Lie algebra $\mathfrak{g}$, there are multiple choices of directions on its Dynkin diagram's edges, thus multiple quivers correspond to the same Lie algebra and the same quantum groups. The qq-characters are sensitive to the directions of edges, also in their $\varepsilon_2 \to 0$ limit.
- Unlike qq-characters, q-characters contain no edge masses $\mathfrak{m}_e$ and no matter polynomials. Thus, we only consider quiver gauge theories with $\underline{m} = 0$ and work in the limit $\mathfrak{m}_e = 0$.
- The qq-characters contain the factors $\mathfrak{q}_i$. We will conjecture how these may contribute to the q-character.

- The S-functions in the qq-characters (cf. Example 4.6) pose no problem, as they converge to 1 in the $\varepsilon_2 \to 0$ limit.

- In the K-theory qq-characters there are also the factors z^{κ_i} and f_i. These have to be forgotten in the $q_2 \to 1$ limit.

- The qq-characters depend on a choice of shift parameters $\underline{\sigma}$ (or $\underline{\tau}$ in K-theory). These have to be chosen correctly so that the limit coincides with the q-characters.

We consider two basic examples in cohomology: The A_1 theory's qq-character is

$$\mathcal{Y}_1(x+\varepsilon+\sigma) + \frac{\mathfrak{q}}{\mathcal{Y}_2(x+\sigma)},$$

which looks no different in the $\varepsilon_2 \to 0$ limit. Compare this with the q-character (97). Clearly, when we follow the *qq-to-q prescription* to

(i) set $x = -\varepsilon$
(ii) flip $\varepsilon \leftrightarrow -\varepsilon$,
(iii) for each monomial $\underline{\mathfrak{q}}^{\underline{k}} \frac{\mathcal{Y}_{i_1} \cdots \mathcal{Y}_{i_a}}{\mathcal{Y}_{j_1} \cdots \mathcal{Y}_{j_b}}$, add $|\underline{k}|\varepsilon$ to the arguments of all $\mathcal{Y}$-functions in the monomial,

the qq-character becomes the q-character. The same prescription works for the first fundamental qq-character of the quiver $1 \leftarrow 2$ (the reversed A_2 quiver; this does not work for the standard A_2 quiver):

$$\mathcal{Y}_1(x+\varepsilon+\sigma) + \mathfrak{q}_1 \frac{\mathcal{Y}_2(x+\varepsilon+\sigma)}{\mathcal{Y}_1(x+\sigma)} + \mathfrak{q}_1\mathfrak{q}_2 \frac{1}{\mathcal{Y}_2(x+\sigma)}$$

Conjecture 4.16. *Let $\mathfrak{g}$ be a simple Lie algebra of type ADE and let U be an irreducible subquotient of the $Y(\mathfrak{g})$-module $U(\underline{\sigma})$. Then there exists a choice of directions on its Dynkin diagram's edges, defining a quiver γ, such that when one applies the qq-to-q prescription, the cohomology qq-character $\mathcal{X}_{\underline{w},\underline{\sigma}}$ of the quiver gauge theory $(\gamma, \underline{n}, 0)$ reduces to the q-character of U in the limit $\varepsilon_2 = 0$.*

In the proof of Lemma 4.13, we found the formula

$$\mathcal{X}_{\underline{w},\underline{\sigma}}(x) = \sum_{\underline{v}} \underline{\mathfrak{q}}^{\underline{v}} \sum_{\eta} \chi(F(\eta)) \prod_{i \in \mathrm{Vert}_\gamma} \left(\prod_{p=1}^{w_i} \mathcal{Y}_i(x+\varepsilon+\sigma_{i,p}) \prod_{t=1}^{v_i} A_i(x+d_{i,t}(\eta))^{-1} \right) \tag{98}$$

for the $\underline{m} = 0$ qq-character in the $\varepsilon_2 \to 0$ limit, with

$$A_i(x) = \frac{\mathcal{Y}_i(x)\mathcal{Y}_i(x+\varepsilon)}{\prod_{e:\mathrm{in}(e)=i} \mathcal{Y}_{\mathrm{out}(e)}(x) \prod_{e:\mathrm{out}(e)=i} \mathcal{Y}_{\mathrm{in}(e)}(x+\varepsilon)}. \tag{99}$$

Clearly, equations (98) and (95) are very similar: In the monomial in the sum they are merely shifted by ε against each other. However, (98) is a sum over fixed point sets of the $T_{\underline{w}} \times T$-action while (95) is over fixed point sets of the A-action. Furthermore, $A_{i,x}$ in (96) and $A_i(x)$ in (99) are different. We do not know how to relate (98) and (95) completely, but believe this to be the key to proving Conjecture 4.16.

There is an obvious extension of this conjecture to K-theory qq-characters and q-characters of $U_q(\mathfrak{g})$; here the qq-to-q prescription includes deleting the z^{κ_i} and f_i from the qq-character.

A standard fact from the theory of q-characters is that, in the $\varepsilon \to 0$ or $q \to 1$ limit, the q-character of an irreducible subquotient of $U(\underline{\sigma})$, $\underline{\sigma} = (\sigma_{i,p})_{i\in\mathrm{Vert}_\gamma, p=1,\dots,w_i}$, (or $V(\underline{\tau})$) reduces to the ordinary character of the irreducible $\mathfrak{g}$-representation with highest weight $\sum_i w_i\Lambda_i$. Thus, a corollary of our conjecture would be that, for ADE quiver gauge theories $(\gamma, \underline{n}, 0)$, the qq-characters of highest weight $\underline{w}$ reduce to the ordinary characters of the associated highest weight representation of $\mathfrak{g}_\gamma$ in the $\vec{\varepsilon}, \underline{\sigma}, \underline{\mathfrak{m}} \to 0$ limit.

Other work on qq-character representation theory. Kimura and Pestun [KP18b] constructed a quiver W-algebra, in which $\mathcal{Y}$-functions are implemented as vertex operators and the fundamental qq-characters are the generating currents.

The connections between quiver gauge theory and representation theory we discussed have been restricted to ADE quivers. We couldn't realize other simple Lie algebras' Dynkin diagrams as quivers as their Cartan matrices are asymmetric, while those of quivers are always symmetric. However, this asymmetric case can be treated in the fractional quiver gauge theories introduced by Kimura and Pestun [KP18a].

Recently, work has been done [Liu22; Bay+23] on the question of finding a purely representation theoretic definition of qq-characters: This is supposed to be the qq version of the original definition of q-characters in terms of the R-matrix of the quantum affine algebra.

5 Conclusion

In the preceding chapters we gave an overview of instanton counting and the nascent theory of qq-characters for quiver gauge theories. Let us review some of the questions we raised.

We expect qq-characters defined by the integrals over quiver varieties to be the result of the recursive desingularization starting with the highest weight monomial of $\mathcal{Y}_i$-functions. To make this precise, one would need to unambiguously provide an algorithm for this process, most likely an adaptation of the Frenkel–Mukhin algorithm. After this, a first step might be to prove the non-singularity of the expectation of the algorithm's result, possibly using the method of Hasse diagrams that we sketched in Example 4.6. Proving that this algorithmic qq-character coincides with the geometric qq-character would likely be more difficult.

Another research direction would be using the non-perturbative Dyson–Schwinger equations for investigating the distributions of the random partitions defined by the Nekrasov partition functions in the limit $\varepsilon_1, \varepsilon_2 \to 0$. The non-perturbative Dyson–Schwinger equations could play a similar role to that of the Dyson–Schwinger equations of random matrix theory. Our Section 4.3 remains at the surface level of what could be possible here. A first analysis of random partitions using non-perturbative Dyson–Schwinger equations was conducted in [BGG17].

For the relationship between q- and qq-characters we provided the Conjecture 4.16 but were unable to prove it. To relate our formula for the $\varepsilon_2 \to 0$ limit of qq-characters 94 to Nakajima's formula for q-characters 95, an analysis of the quiver varieties' fixed point submanifolds under the actions by $T_{\underline{w}} \times T$ and A, as well as the structures of the isotropy representations of the bundles $\mathbf{V}_i$ is needed.

Recently, much research on qq-characters has been focused on finding a purely representation theoretic definition or interpretation of them. For the case of A-type quivers, [Bay+23] presents a definition based on R-matrices, similar to the original definition of q-characters [FR99]. Extending this definition to more general quivers remains an open problem.

Appendix A: Partitions and Young Diagrams

Definition A.1. *Let k be a non-negative integer. A partition of k is a finite, non-ascending sequence of positive integers*

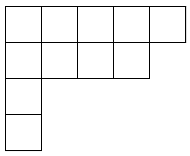
$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L > 0)$$

that sums to $k = \sum_{s=1}^{L} \lambda_s$.

Here $L = \mathrm{length}(\lambda)$ is called the length of the partition and k is also called the partition's size, denoted $|\lambda|$. We denote the set of all partitions of size k by $\mathfrak{P}(k)$. To each partition λ we associate a Young diagram. This is a diagram of boxes, each with a vertical coordinate $s \in \mathbb{Z}_{>0}$ and a horizontal coordinate $t \in \mathbb{Z}_{>0}$. The set of boxes in the Young diagram associated to λ is

$$\{(s_1, s_2) \in \mathbb{Z}_{>0}^2 \mid s_1 = 1, \ldots, L \text{ and } s_2 = 1, \ldots, \lambda_s\}.$$

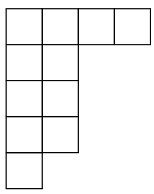
The partition's size is exactly the number of boxes in the Young diagram. As an example, consider the Young diagram of the partition $\lambda = (5, 4, 1, 1)$.



To a partition λ, we associate the transpose partition $\check{\lambda}$, defined by

$$\check{\lambda}_s = \#\{i \in \mathbb{Z}_{>0} \mid \lambda_i \geq s\}.$$

The transpose partition's Young diagram is exactly the transpose of the original partition's Young diagram. As an example, for the partition $\lambda = (5, 4, 1, 1)$, whose Young diagram we considered above, the transpose is $\check{\lambda} = (4, 2, 2, 2, 1)$ and its Young diagram is



It is clear that, while the length of a partition's transpose may be different from its own length, their sizes are the same.

We will generally identify partitions with their Young diagrams.

Definition A.2. *Let $n \geq 1$ and $k \geq 0$ be integers. An n-colored partition of k is an n-tuple of partitions*

$$\boldsymbol{\lambda} = (\lambda^{\alpha})_{\alpha=1,\ldots,n}$$

whose sizes sum to $k = \sum_{\alpha=1}^{n} |\lambda^{\alpha}|$.

We denote the set of all n-colored partitions of size k by $\mathfrak{P}(n, k)$. Of course, to such an n-colored partition are associated n Young diagrams in the fashion outlined above. We write

$$k[\boldsymbol{\lambda}] \coloneqq \sum_{\alpha=1}^{n} |\lambda^\alpha|$$

We sometimes write $(\alpha, \square) \in \boldsymbol{\lambda}$ for $\alpha = 1, \ldots, n$ and $\square \in \lambda^\alpha$. We call $(\alpha, \square)$ a box in $\boldsymbol{\lambda}$.

In the setting of quiver gauge theories, a quiver is colored by associating to each vertex i a positive integer n_i.

Definition A.3. *Let $\underline{n} = (n_i)_{i \in \mathrm{Vert}_\gamma} \in \mathbb{Z}_{>0}^{\mathrm{Vert}_\gamma}$ be a coloring of a quiver γ. An $\underline{n}$-colored partition of $\underline{k} = (k_i)_{i \in \mathrm{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$ is an assignment*

$$\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^i)_{i \in \mathrm{Vert}_\gamma}$$

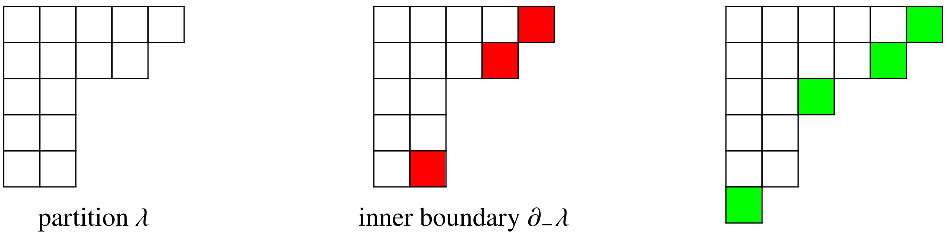
of an n_i-colored partition $\boldsymbol{\lambda}^i = (\lambda^{i\alpha})_{\alpha=1,\ldots,n_i}$ of size k_i to each vertex i.

We denote the set of all $\underline{n}$-colored partitions of $\underline{k}$ for a quiver γ by $\mathfrak{P}(\underline{n}, \underline{k})$. We write

$$\underline{k}[\underline{\boldsymbol{\lambda}}] \coloneqq \left(k_i[\underline{\boldsymbol{\lambda}}]\right)_{i \in \mathrm{Vert}_\gamma} \coloneqq \left(k[\boldsymbol{\lambda}^i]\right)_{i \in \mathrm{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}.$$

We sometimes write $(i, \alpha, \square) \in \underline{\boldsymbol{\lambda}}$ for $i \in \mathrm{Vert}_\gamma$, $\alpha = 1, \ldots, n$, and $\square \in \lambda^{i\alpha}$. We call $(i, \alpha, \square)$ a box in $\underline{\boldsymbol{\lambda}}$.

Outer and inner boundaries. For a partition's Young diagram, we distinguish between the outer and the inner boundary. The outer boundary $\partial_+\lambda$ consists of all boxes $(s_1, s_2) \in \mathbb{Z}_{>0}^2$ outside the Young diagram such that the boxes $(s_1 - 1, s_2)$ and $(s_1, s_2 - 1)$ (left and above (s_1, s_2)) are contained in the Young diagram or such that one of these boxes is in the Young diagram and that (s_1, s_2) lies in the first row ($s_1 = 1$) or first column ($s_2 = 1$). The inner boundary $\partial_-\lambda$ consists of all boxes $(s_1, s_2) \in \mathbb{Z}_{>0}^2$ inside the Young diagram such that the boxes below (box $(s_1 + 1, s_2)$) and to their right (box $(s_1, s_2 + 1)$) are outside the Young diagram. We illustrate these concepts for the partition $\lambda = (5, 4, 2, 2, 2)$:



partition λ | inner boundary $\partial_-\lambda$ | outer boundary $\partial_+\lambda$

Note that for all partitions λ there is one box more in the outer than in the inner boundary, i.e. $\#\partial_+\lambda - \#\partial_-\lambda = 1$. The significance of these boundaries for non-perturbative Dyson–Schwinger equations is that if one wants to increase the size of a partition by 1, one has to do this by adding a box in the outer boundary to the Young diagram, thus turning it into an inner boundary. If one wants to decrease the size, one has to take away a box from the inner boundary, thus turning it into an outer boundary.

Torus representations from partitions. A partition λ of size k defines an ideal $I_\lambda \subset \mathbb{C}[X, Y]$ generated by the monomials $x^{s_1 - 1} y^{s_2 - 1}$ for each box $(s_1, s_2) \in \mathbb{Z}_{>0}^2$ outside the partition. A smaller generating set consists of the monomials $x^{s_1-1}y^{s_2-1}$ for each box $(s_1, s_2) \in \partial_+\lambda$ in the outer boundary of λ. The quotient

$$K_\lambda = \mathbb{C}[x, y]/I_\lambda$$

is a complex vector space of dimension k. The torus $(\mathbb{C}^\times)^2$ is represented on $\mathbb{C}[x, y]$ by

$$(q_1, q_2) \cdot x^{s_1-1}y^{s_2-1} = (q_1x_1)^{s_1-1}(q_2x_2)^{s_2-1} = e^{c(s_1,s_2)}x^{s_1-1}y^{s_2-1},$$

where $q_1 = e^{\varepsilon_1}$, $q_2 = e^{\varepsilon_2}$, and

$$c_\square \coloneqq (s_1 - 1)\varepsilon_1 + (s_2 - 1)\varepsilon_2$$

is called the content of the box $\square = (s_1, s_2)$. As the ideal I_λ is an invariant subspace of this representation, the quotient K_λ is a representation of $(\mathbb{C}^\times)^2$ too. Its character is

$$\chi(K_\lambda) = \sum_{\square\in\lambda} e^{c_\square}.$$

We denote by $e^{\mathfrak{a}_\alpha}$, $\alpha = 1, \dots, n$ the 1-dimensional representation of $(\mathbb{C}^\times)^n$ defined by $(e^{\mathfrak{a}_1}, \dots, e^{\mathfrak{a}_n}) \mapsto e^{\mathfrak{a}_\alpha}$, i.e. the fundamental representation of the α-th factor group. An n-colored partition $\boldsymbol{\lambda}$ then defines a representation of $(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^2$, the complexification of the maximal torus in $U(n) \times \mathrm{Spin}(4)$,

$$K_{\boldsymbol{\lambda}} = \sum_{\alpha=1}^{n} \left(e^{\mathfrak{a}_\alpha} \sum_{\square\in\lambda^\alpha} K_\lambda \right)$$

of dimension $k[\boldsymbol{\lambda}]$. Its character is

$$\chi(K_{\boldsymbol{\lambda}}) = \sum_{(\alpha,\square)\in\boldsymbol{\lambda}} e^{\mathfrak{a}_\alpha + c_\square}.$$

If we have a quiver and an $\underline{n}$-colored partition $\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^i)_{i\in\mathrm{Vert}_\gamma}$, then for each vertex $i \in \mathrm{Vert}_\gamma$ there is the representation

$$K_i[\underline{\boldsymbol{\lambda}}] \coloneqq K_{\boldsymbol{\lambda}^i},$$

used in the derivation of the non-perturbative transformation of the measure.

Appendix B: Dyson–Schwinger equations

Dyson–Schwinger equations in quantum field theory (QFT) and random matrix theory (RMT) are essentially just integration by parts formulas. We discuss how integration by parts formulas are a consequence of translational symmetry of the integration measure (more generally; parametrization invariance of the integral). This perspective of Dyson–Schwinger equations originating from the measure's translation-invariance is important for us because the non-perturbative Dyson–Schwinger equations of $\mathcal{N} = 2$ supersymmetric quiver gauge theories (cf. Chapter 4) originate from a similar feature of a measure on a discrete set.

Suppose V is a vector space (e.g. the infinite-dimensional space of classical fields for a QFT, or a space of matrices in RMT) together with a translation-invariant measure μ (which is problematic in the infinite-dimensional situation of path integrals, but this is generally ignored in physics):

$$\mu(A + \epsilon) = \mu(A)$$

where A is some measurable set in V and $\epsilon \in V$ some translation vector. We define the functional

$$I(\epsilon) := \int_V f(v + \epsilon)g(v)\, d\mu(v) = \int_V f(v)g(v - \epsilon)\, d\mu(v)$$

by translation-invariance. Assuming differentiation with respect to ϵ and integration are interchangable,

$$DI(0)\epsilon = \int_V (Df(v)\epsilon)g(v)\, d\mu(v) = -\int_V f(v)(Dg(v)\epsilon)\, d\mu(v),$$

which is the integration by parts formula with vanishing boundary terms.

Building upon this derivation of integration by parts from translational symmetry, we discuss (perturbative) Dyson–Schwinger equations in quantum field theory and random matrix theory, as well as some applications.

B.1 Dyson–Schwinger equations in quantum field theory

In the path integral formalism of quantum field theory, the action $S : \mathcal{F} \to \mathbb{R}$ on the space of classical fields $\mathcal{F}$ (which one should think of as a vector space of functions on spacetime $\mathbb{R}^4$) defines the partition function

$$\mathcal{Z} = \int_{\mathcal{F}} [D\phi] e^{-S[\phi]}$$

as the integral over the infinite-dimensional space $\mathcal{F}$ of the functional e^{-S} (this is after Wick rotation, without Wick rotation one instead integrates e^{iS}). While this integral is mathematically problematic to define, it has proved useful in physics to proceed as if it existed and had the standard properties of integrals. One may now think of $\frac{e^{-S}}{\mathcal{Z}}$ as a kind of probability measure on the field space $\mathcal{F}$. In this formalism, observables are functionals $\mathcal{O} : \mathcal{F} \to \mathbb{R}$ and may thus be integrated with respect to the aforementioned measure, yielding their expectations

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{F}} [D\phi] e^{-S[\phi]} \mathcal{O}[\phi]$$

Assuming translation-invariance of the path integral measure and vanishing boundary terms, we can perform integration by parts for the functional $e^{-S}\mathcal{O}$ (where we use functional derivatives $\delta/\delta\phi(x)$):

$$\int [D\phi]e^{-S[\phi]}\frac{\delta S[\phi]}{\delta\phi(x)}\mathcal{O}[\phi] = \int [D\phi]e^{-S[\phi]}\frac{\delta\mathcal{O}[\phi]}{\delta\phi(x)},$$

or in the probabilistic notation,

$$\left\langle \frac{\delta S}{\delta\phi(x)}\mathcal{O}\right\rangle = \left\langle \frac{\delta\mathcal{O}}{\delta\phi(x)}\right\rangle. \tag{100}$$

Typically, one considers Dyson–Schwinger equations for the observables

$$\mathcal{O}_J[\phi] = e^{\int d^4x J(x)\phi(x)}$$

where $J : \mathbb{R}^4 \longrightarrow \mathbb{R}$ (for real scalar field theories) is a source field. The significance of these observables is that the QFT's correlation functions (Green's functions) can be obtained from them: Define the generating functional

$$Z[J] = \int [D\phi]e^{-S[\phi]+\int d^4xJ(x)\phi(x)} = \mathcal{Z}\langle\mathcal{O}_J\rangle .$$

Then the n-point correlation function $G_n(x_1,\dots,x_n)$, which is the expectation $\langle\phi(x_1)\dots\phi(x_n)\rangle$, satisfies

$$G_n(x_1,\dots,x_n) = \frac{1}{\mathcal{Z}}\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\Big|_{J=0}.$$

The Dyson–Schwinger equation for $\mathcal{O}_J$ is

$$\left\langle \frac{\delta S}{\delta\phi(x)}\mathcal{O}_J\right\rangle = \langle J(x)\mathcal{O}_J\rangle$$

and, using that $\mathcal{Z}\langle\phi^n\mathcal{O}_J\rangle = \frac{\delta^n}{\delta J^n}Z[J]$, it can be seen that

$$\mathcal{Z}\left\langle \frac{\delta S}{\delta\phi}\mathcal{O}_J\right\rangle = \frac{\delta S}{\delta\phi}\left[\frac{\delta}{\delta J}\right]Z[J],$$

thus the Dyson–Schwinger equation is

$$\frac{\delta S}{\delta\phi}\left[\frac{\delta}{\delta J}\right]Z[J] - JZ[J] = 0. \tag{101}$$

For the example of ϕ^4 theory where the Lagrangian is $\mathcal{L} = \frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] - \frac{g}{4!}\phi^4$, using $\frac{\delta S}{\delta\phi} = -\partial^2\phi - m^2\phi - \frac{g}{6}\phi^3$, the Dyson–Schwinger equation (101) is

$$\left[\partial^2\frac{\delta}{\delta J(x)} + m^2\frac{\delta}{\delta J(x)} + \frac{g}{6}\frac{\delta^3}{\delta J(x)^3}\right]Z[J] + JZ[J] = 0$$

Now we can successively take infinitely many functional derivatives ($\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}$) of this equation. Each of the infinitely many equations obtained is evaluated at $J = 0$, making it an equation for the correlation functions G_n:

$$(\partial_x^2 + m^2)G_1(x) + \frac{g}{6}G_3(x,x,x) = 0,$$

$$(\partial_x^2 + m^2)G_2(x,x_1) + \frac{g}{6}G_4(x,x,x,x_1) + \delta(x-x_1)G_1(x) = 0,$$
$$(\partial_x^2 + m^2)G_3(x,x_1,x_2) + \frac{g}{6}G_5(x,x,x,x_1,x_2) + \sum_{i=1}^{2}\delta(x-x_i)G_2(x,x_i) = 0,$$
$$\dots$$

B.2 Dyson–Schwinger equations in random matrix theory

Here, we replace the field ϕ by a matrix M, meaning we pass from an infinite-dimensional to a finite-dimensional setting which has the advantage of integrals being well-defined so that the derivation of Dyson–Schwinger equations is rigorous. In RMT, Dyson–Schwinger equations are also called loop equations. Our treatment is informal and follows that of Eynard [EKR15, Sec. 4.1]. Consider an ensemble H_N of Hermitian $N \times N$ matrices. The probability distribution is specified by the action

$$S(M) = N \operatorname{tr} V(M), \qquad V(x) \in \mathbb{R}[x]$$

given by a polynomial potential. Then the partition function is

$$Z = \int_{H_N} e^{-N \operatorname{tr} V(M)} dM$$

where dM is the Lebesgue measure.

The RMT version of the Dyson–Schwinger equation (100) is $\left\langle \frac{\partial S}{\partial M_{ij}} \mathcal{O} \right\rangle = \left\langle \frac{\partial \mathcal{O}}{\partial M_{ij}} \right\rangle$ where $\mathcal{O}$ is some function $H_N \to \mathbb{C}$. We consider the observables $\mathcal{O}_{ij}[M] = (M^k)_{ij}$. From the relations

$$\frac{\partial}{\partial M_{ij}} \operatorname{tr} V(M) = V'(M)_{ji} \qquad \text{and} \qquad \frac{\partial}{\partial M_{ij}} (M^k)_{ij} = \sum_{l=0}^{k-1} (M^l)_{ii} (M^{k-l-1})_{jj},$$

it becomes

$$N \left\langle (M^k)_{ij} V'(M)_{ji} \right\rangle = \sum_{l=0}^{k-1} \left\langle (M^l)_{ii} (M^{k-l-1})_{jj} \right\rangle .$$

Summing this relation over i, j yields

$$N \left\langle \operatorname{tr}\left(M^k V'(M)\right) \right\rangle = \sum_{l=0}^{k-1} \left\langle \operatorname{tr} M^k \operatorname{tr} M^{k-l-1} \right\rangle .$$

In order to encode all these equations into a single functional relation, we add these equations over k with a factor of x^{-k-1}, where the variable x is considered formal:

$$\begin{aligned}
& N \sum_{k=0}^{\infty} x^{-k-1} \left\langle \operatorname{tr}(M^k V'(M)) \right\rangle = \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} x^{-k-1} \left\langle \operatorname{tr} M^l \operatorname{tr} M^{k-l-1} \right\rangle \\
\iff \quad & N \left\langle \operatorname{tr}\left(\Big(\sum_{k=0}^{\infty} x^{-k-1} M^k \Big) V'(M) \right) \right\rangle = \sum_{a,b=0}^{\infty} x^{-a-b-2} \left\langle \operatorname{tr} M^a \operatorname{tr} M^b \right\rangle \\
\iff \quad & \frac{1}{N} \left\langle \operatorname{tr}\left(\frac{1}{x-M} V'(M) \right) \right\rangle = \frac{1}{N^2} \left\langle \operatorname{tr} \frac{1}{x-M} \operatorname{tr} \frac{1}{x-M} \right\rangle, \qquad (102)
\end{aligned}$$

where we use the matrix resolvent

$$\frac{1}{x-M} = \sum_{k\geq 0} x^{-k-1} M^k.$$

Equation (102) can be expressed in terms of the one-point and two-point correlation functions

$$W_1(x) = \frac{1}{N}\left\langle \operatorname{tr}\frac{1}{x-M}\right\rangle, \qquad W_2(x_1,x_2) = \frac{1}{N^2}\left\langle \operatorname{tr}\frac{1}{x_1-M}\operatorname{tr}\frac{1}{x_2-M}\right\rangle - W_1(x_1)W_1(x_2)$$

as

$$W_2(x,x) + W_1(x)^2 - V'(x)W_1(x) = -P(x) \tag{103}$$

where $P(x) = \frac{1}{N}\left\langle \operatorname{tr}\frac{V'(x)-V'(M)}{x-M}\right\rangle$ is a polynomial.

Growing the size N of the matrix to infinity, the limit

$$\overline{W}(x) \coloneqq \lim_{N\to\infty}\frac{1}{N}\operatorname{tr}\frac{1}{x-M}$$

is deterministic as, in the limit, the probability distribution concentrates at the minimizer of the action. The large-N limit of the Dyson–Schwinger equation (103) is

$$\overline{W}(x)^2 - V'(x)\overline{W}(x) = -\overline{P}(x) \tag{104}$$

where $\overline{P}(x) = \lim_{N\to\infty}\frac{1}{N}\operatorname{tr}\frac{V'(x)-V'(M)}{x-M}$ is polynomial. This is an algebraic equation for $\overline{W}(x)$ and the equation $y^2 - V'(x)y + P(x) = 0$ defines an algebraic curve, called the spectral curve of the matrix model. While $\overline{W}(x)$ has branch cuts or can be regarded as a multi-valued function defined by (104), as a function on the spectral curve it is analytic and single-valued.

The spectral curve encodes much information about the matrix model. In particular, in settings where there's a topological expansion (for example those of [BG13])

$$W_n = \sum_{g\geq 0} N^{2-2g-n} W_{g,n},$$

the correlation functions W_n (as well as the free energy $F = W_0 = -\log Z$) can be computed from $W_{0,1}$ and $W_{0,2}$ by topological recursion (cf. [Bou24; Eyn+16]), using residues at the branch points of the curve.

Under suitable assumptions, the eigenvalues of M are located at the poles of the resolvent $\frac{1}{x-M}$, so that the equilibrium density of eigenvalues

$$\overline{\rho}(x) \coloneqq \lim_{N\to\infty}\left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right\rangle$$

(λ_i are the random eigenvalues of M) is supported on the cuts of $\overline{W}(x)$ [EKR15, Sec. 3.2]. In fact,

$$\overline{\rho}(x) = \frac{1}{\pi}\operatorname{Im}\overline{W}(x+i0).$$

To this thesis, the relevance of the RMT Dyson–Schwinger equations' use in investigating the large-N limit of matrix models is that the non-perturbative Dyson–Schwinger equations can be used in a similar way to investigate the measure, defined by the Nekrasov partition function on the space of partitions, in the limit $\varepsilon_1, \varepsilon_2 \to 0$. We briefly explore this in Section 4.3.

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig verfasst und noch nicht für andere Prüfungen eingereicht habe. Sämtliche Quellen, einschließlich Internetquellen, die unverändert oder abgewandelt wiedergegeben werden, insbesondere Quellen für Texte, Grafiken, Tabellen und Bilder, sind als solche kenntlich gemacht. Mir ist bekannt, dass bei Verstößen gegen diese Grundsätze ein Verfahren wegen Täuschungsversuchs bzw. Täuschung eingeleitet wird.

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